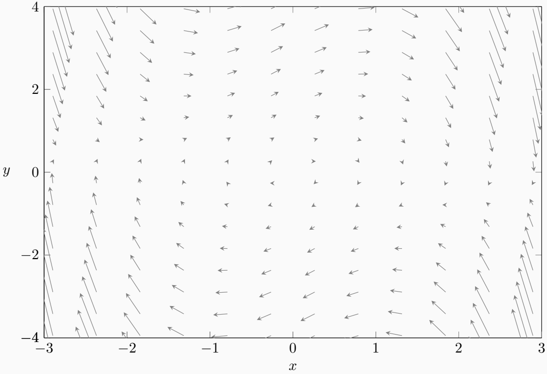
Polynomial Vector Fields in the Plane

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases} \qquad P, Q \in \mathbb{R}[x,y]$$



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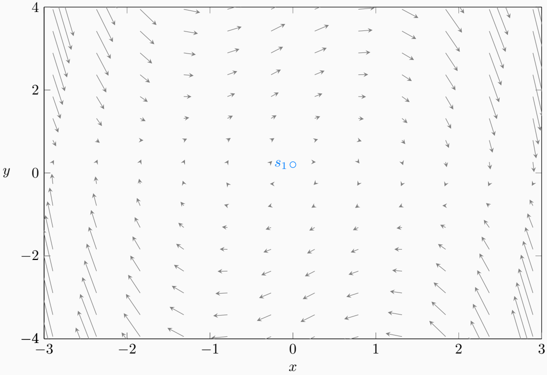

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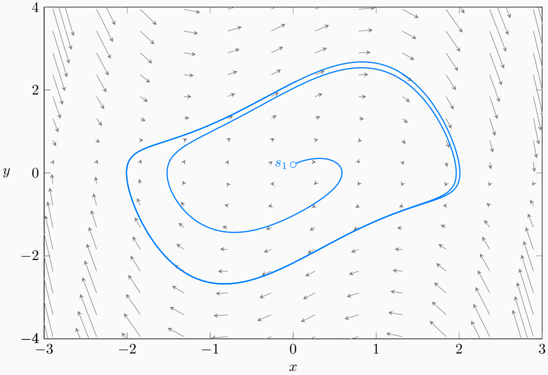

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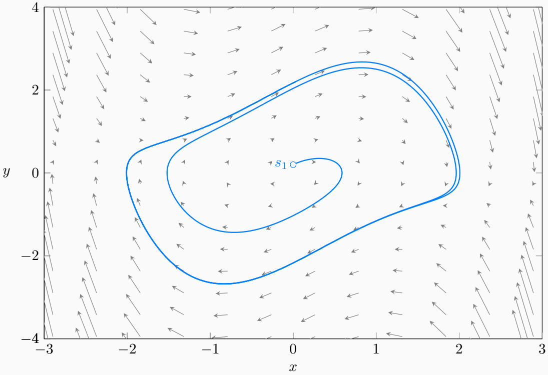

Examples:

- $(v, a) = (\dot{x}, \dot{v}) = (v, Q(x, v))$ in mechanics
- $(\dot{u}, \dot{i}) = (P(u, i), Q(u, i))$ in electricity

Limit Cycles

limit cycle = isolated periodic orbit

(For example, $(\dot{x}, \dot{y}) = (-y, x)$ produces a continuum of periodic orbits but no limit cycles!)

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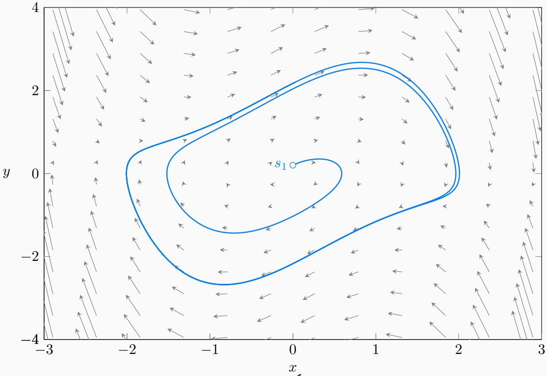

Van der Pol oscillator $$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1 - x^2)y - x \end{cases}$$

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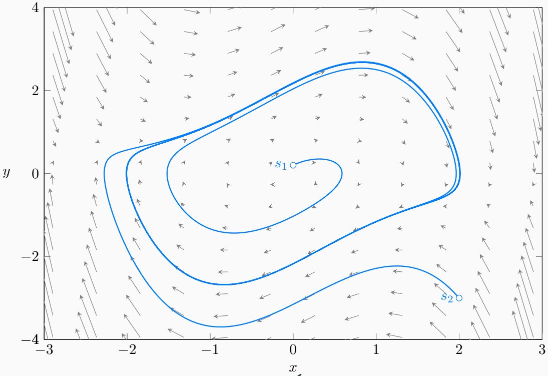

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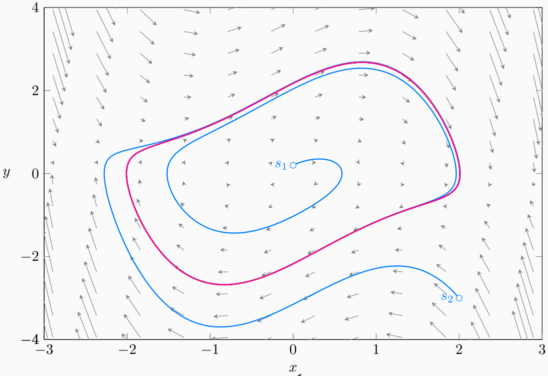

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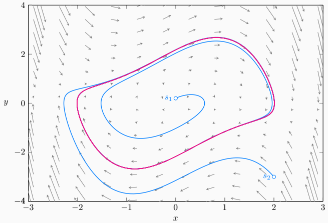
Hilbert's 16th problem (second part)

For a given integer n, what is the maximum number $\mathcal{H}(n)$ of limit cycles a polynomial vector field of degree at most n in the plane can have?

D. Hilbert, International Congress of Mathematicians, Paris, 1900



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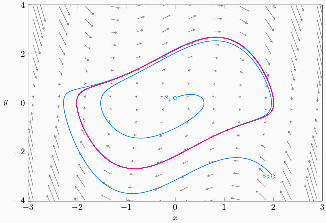

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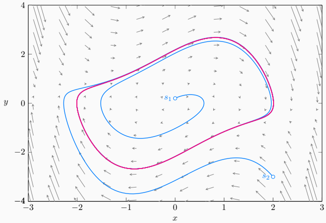
- Some lower bounds: $\mathcal{H}(2) \geq 4$, $\mathcal{H}(3) \geq 13$, $\mathcal{H}(4) \geq 28$.



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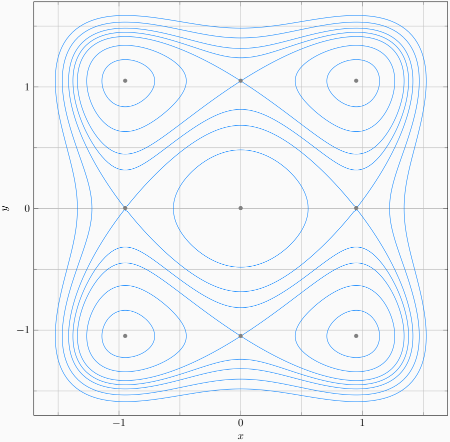

$$H(x, y) = (x^2 - 0.9)^2 + (y^2 - 1.1)^2$$

T. Johnson, A quartic system with twenty-six limit cycles, *Experimental Mathematics*, 2011

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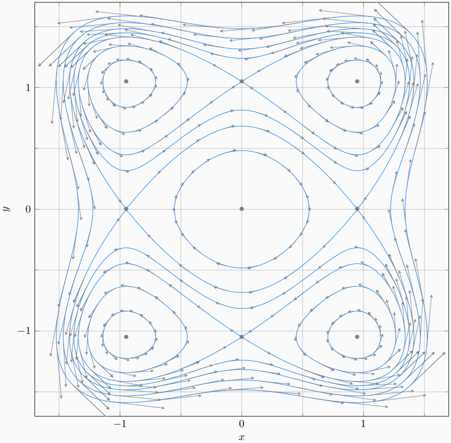

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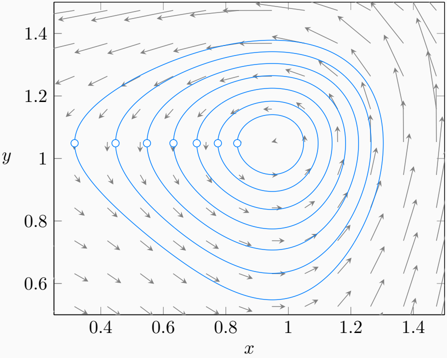

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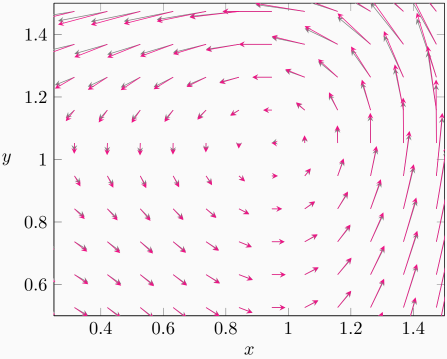

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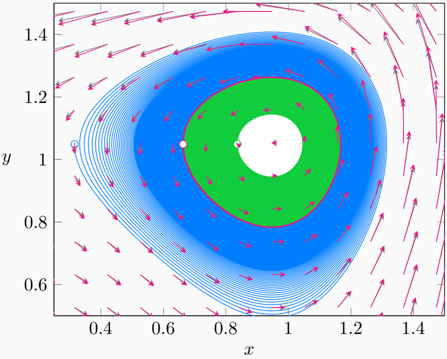

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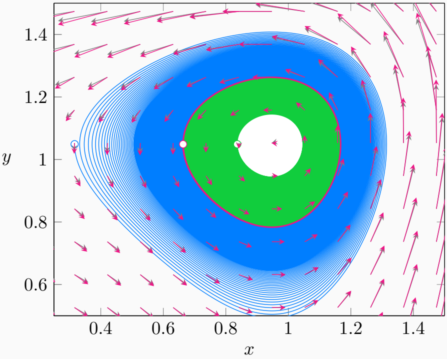

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Infinitesimal Hilbert's 16th problem

For a given integer n, what is the maximal number $\mathcal{Z}(n)$ of limit cycles a perturbed Hamiltonian vector field of the form:

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can have when $\varepsilon \to 0$, with:

- $H(x,y)$ a polynomial potential function of degree $n+1$
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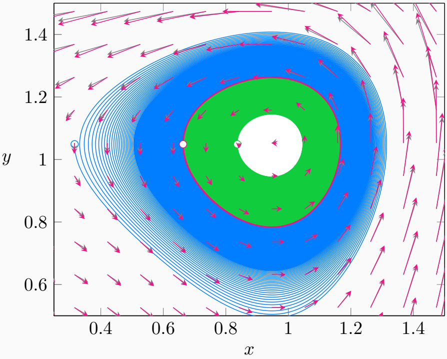

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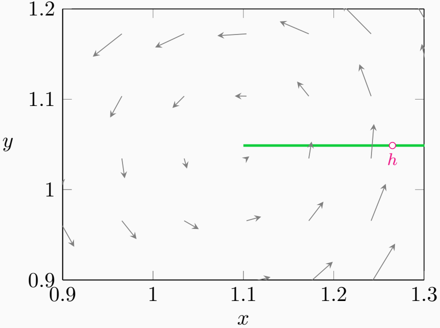
- $\mathcal{Z}(n) < \infty$ for all n
- Pessimistic upper bounds

A Fundamental Tool: the Poincaré-Pontryagin Theorem



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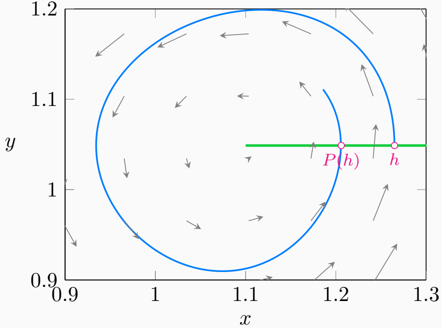

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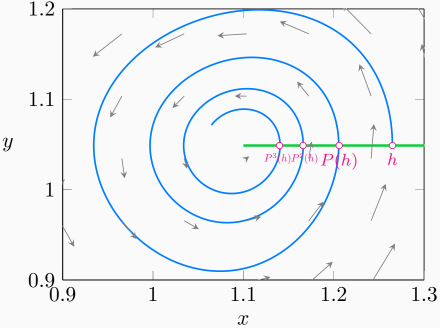

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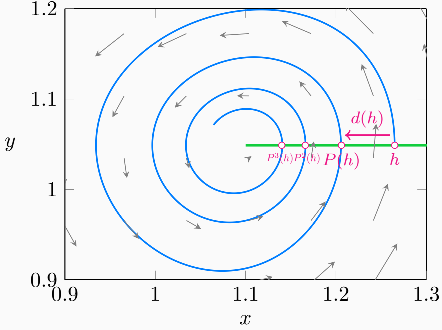

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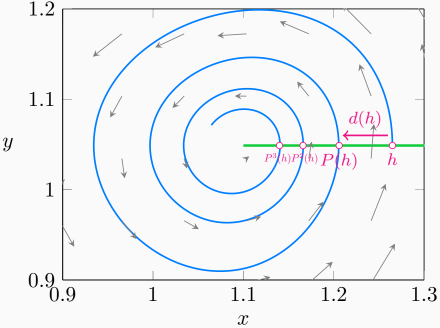

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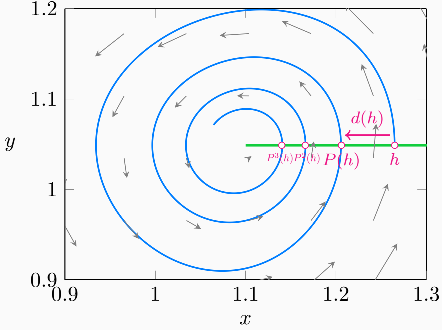

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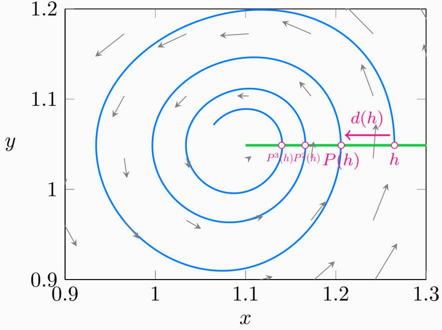

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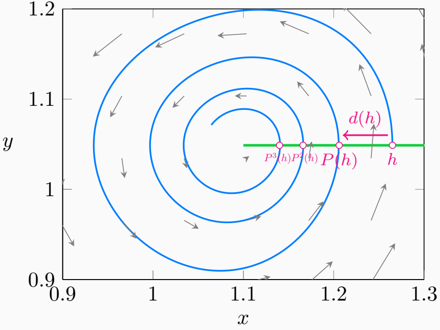

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limit cycles $\equiv$ changes of sign of $\mathcal{I}(h)$ $\equiv$ simple zeros of $\mathcal{I}(h)$

A Pseudo-Hamiltonian Quartic System

- Hamiltonian system:

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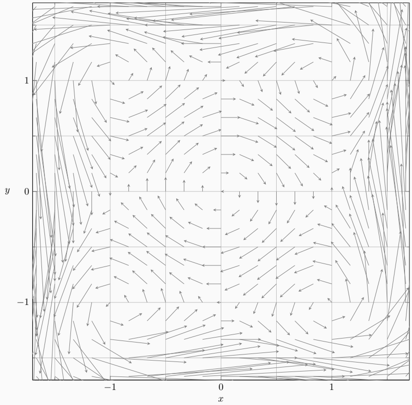

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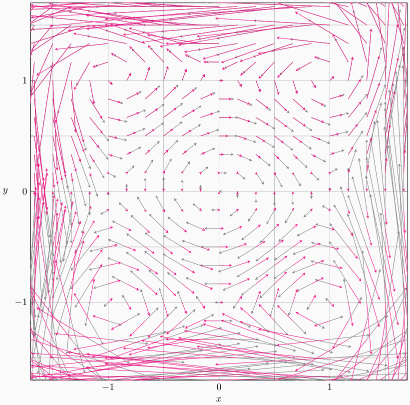

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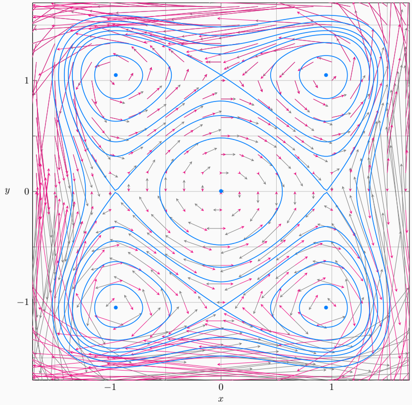

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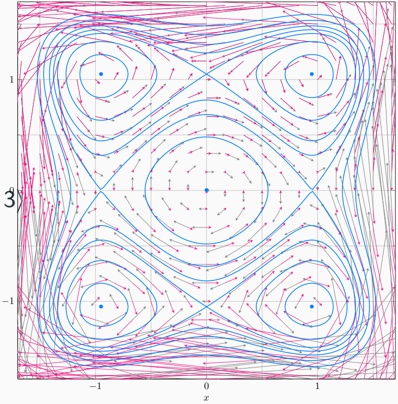

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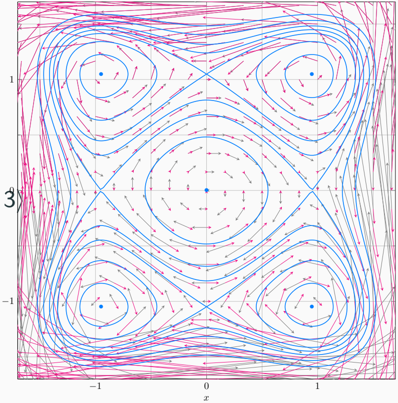

Generalized Poincaré-Pontryagin theorem

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$$\mathcal{I}(h) = \oint_{H^{-1}(h)} \frac{f(x,y)\mathrm{d}y - g(x,y)\mathrm{d}x}{y}$$

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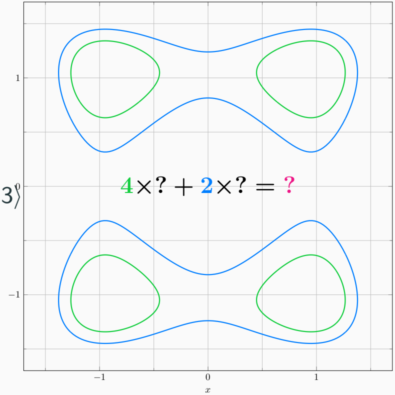

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$\Rightarrow$ Maximize the number of zeros of $\mathcal{I}(h)$

polynomial or rational functions

$$\int_{\Gamma(h)} P(x,y)\mathrm{d}y - Q(x,y)\mathrm{d}x$$

oval $H(x,y) = h$ with H polynomial or rational

Challenge for Abelian Integrals

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- Find the coefficients of P and Q
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Challenge for Abelian Integrals

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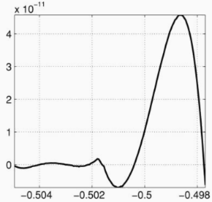

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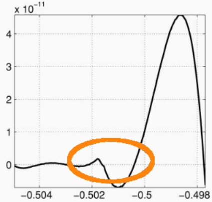

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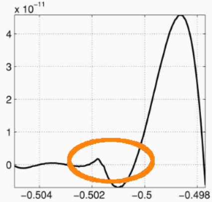
- Certified implementation in a proof assistant
  e.g., T. Johnson, 2011: $\mathcal{H}(4) \geq$ ~~26~~ 22 $\rightarrow$ 24
  (Bréhard, Brisebarre, Joldes, Tucker)



Credit: T. Johnson

Choice of Perturbations

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f(x,y) & = & 1 & x & y & x^2 & xy \\
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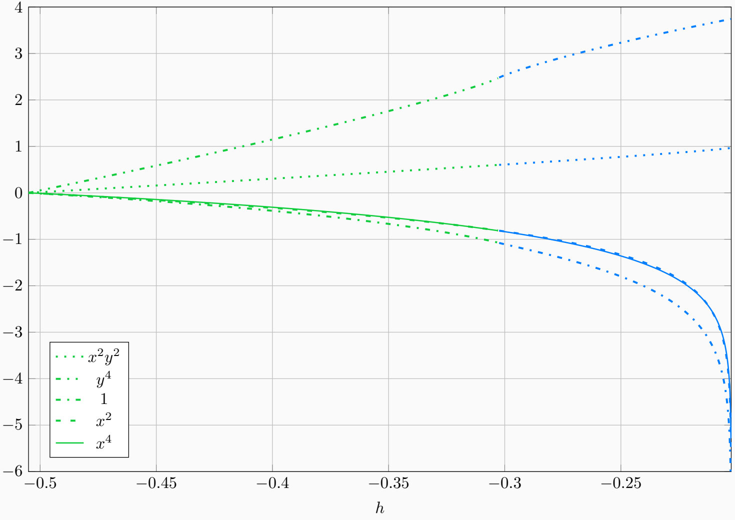
$$\mathcal{I}(h) = \oint_{H^{-1}(h)} \frac{\alpha_{00} + \alpha_{20}x^2 + \alpha_{22}x^2y^2 + \alpha_{40}x^4 + \alpha_{04}y^4}{y} \mathrm{d}x$$

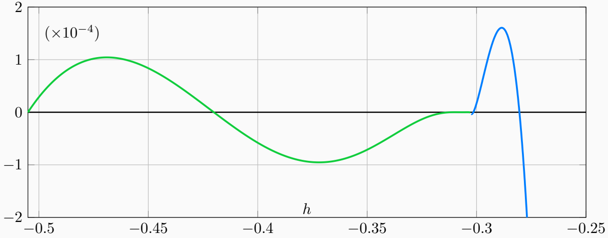
Numerically Optimizing the Number of Zeros

- Find coefficients of $\mathcal{I}(h) = \alpha_{00}\mathcal{I}_{00}(h) + \alpha_{20}\mathcal{I}_{20}(h) + \alpha_{22}\mathcal{I}_{22}(h) + \alpha_{40}\mathcal{I}_{40}(h) + \alpha_{04}\mathcal{I}_{04}(h$



Numerically Optimizing the Number of Zeros

α_{00}	=	-0.78622148667854837664
α_{20}	=	0.87723523612653436051
α_{22}	=	1
α_{40}	=	0.23742713894293038223
α_{04}	=	-0.21823846173078863753



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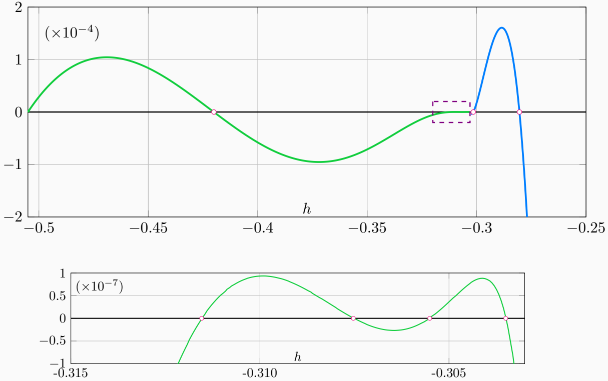

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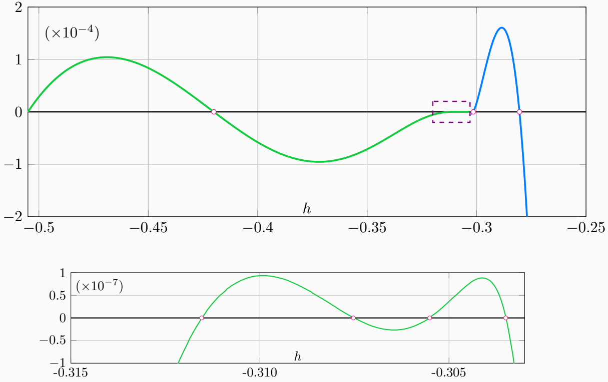
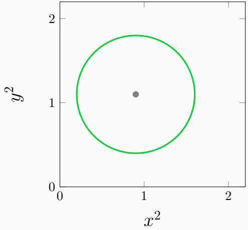


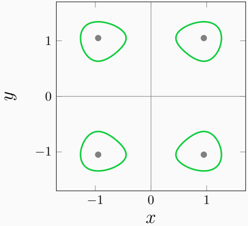
$4 \times 5 + 2 \times 2 = 24$

Computing Abelian Integrals

$$H(x, y) = (x^2 - 0.9)^2 + (y^2 - 1.1)^2$$

$$0 < \sqrt{h} =: r < 0.9$$





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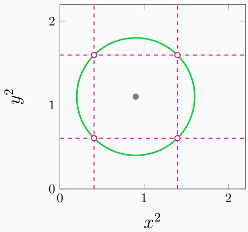
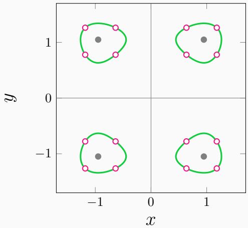


$$x_{\text{min}} = \sqrt{0.9 - \frac{r}{\sqrt{2}}} \qquad x_{\text{max}} = \sqrt{0.9 + \frac{r}{\sqrt{2}}}$$

$$y_{\text{min}} = \sqrt{1.1 - \frac{r}{\sqrt{2}}} \qquad y_{\text{max}} = \sqrt{1.1 + \frac{r}{\sqrt{2}}}$$



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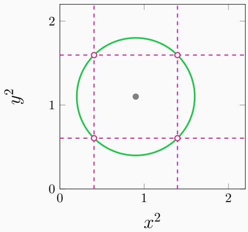

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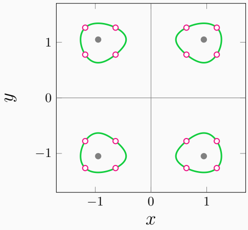

$$y_{\text{up}}(x) = \sqrt{1.1 + \sqrt{r^2 - (x^2 - 0.9)^2}}$$

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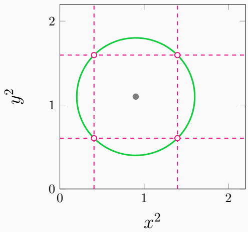
$$H(x,y) = (x^2-0.9)^2+(y^2-1.1)^2 \qquad 0 < \sqrt{h} =: r < 0.9$$



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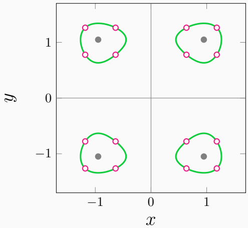


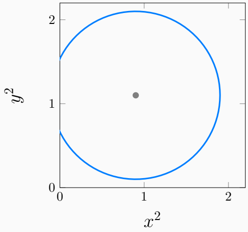
$$\mathcal{I}(h) = \oint_{H^{-1}(h)} \frac{g(x,y)}{y}\,\mathrm{d}x = \int_{x_{\mathbf{min}}}^{x_{\mathbf{max}}} \left(\frac{g(x, y_{\mathbf{up}}(x))}{y_{\mathbf{up}}(x)} - \frac{g(x, y_{\mathbf{down}}(x))}{y_{\mathbf{down}}(x)} \right) \mathrm{d}x$$

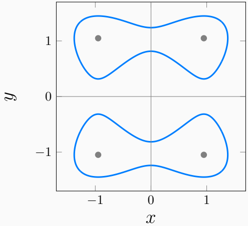
$$+ \int_{y_{\mathbf{min}}}^{y_{\mathbf{max}}} \left(\frac{g(x_{\mathbf{left}}(y), y)}{x_{\mathbf{left}}(y)} + \frac{g(x_{\mathbf{right}}(y), y)}{x_{\mathbf{right}}(y)} \right) \frac{y^2 - 1.1}{\sqrt{r^2 - (y^2 - 1.1)^2}}\,\mathrm{d}y.$$

Computing Abelian Integrals

$H(x, y) = (x^2 - 0.9)^2 + (y^2 - 1.1)^2$

$0.9 < \sqrt{h} =: r < 1.1$





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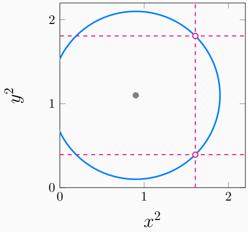

$$x_{\text{min}} = 0 \qquad x_{\text{max}} = \sqrt{0.9 + \frac{r}{\sqrt{2}}}$$

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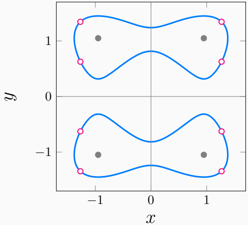

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$$+ 2 \int_{y_{\text{min}}}^{y_{\text{max}}} \frac{g(x_{\text{right}}(y), y)(y^2 - 1.1)}{x_{\text{right}}(y)\sqrt{r^2 - (y^2 - 1.1)^2}} \mathrm{d}y.$$