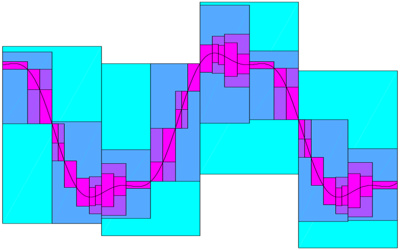
Validated Numerics
Some chaotic bits and pieces



M. Joldes

October 19, 2023

Joined works with F. Bréhard, N. Brisebarre, W. Tucker

Outline

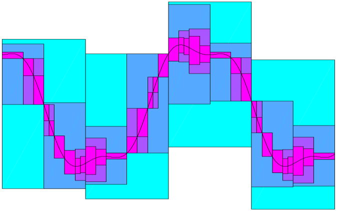
Can/Should we trust the numerics?

- Floating-Point Arithmetic
- Validated Computing with Interval Arithmetic
- Rigorous Polynomial Approximations
- A posteriori error bounds with Newton-like methods
- Applications:
  - Computer-aided proof for the existence of sinks in the Henon map
  - Validated approximation of solutions of Newton-like operators

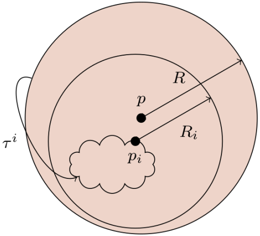
Validated Computing Strategies

Two strategies:

- Iterative refinement of enclosures
- *All* computations are validated



- Numerical solution easily available
- Existence + Explicit error bounds for a *true* solution ⇝ fixed-point arguments
- A Posteriori Newton-like Validation



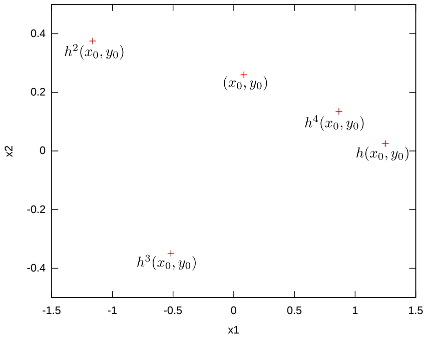
Part 1: Computer-Assisted Proof for Finding Sinks of the Henon Map

Hénon attractor

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - a x_1^2, b x_1)$$

- Map iterations: $h_{a,b}^{i+1} := h_{a,b} \circ h_{a,b}^{i}, i \in \mathbb{N}^*$.
- For classical parameter values $a = 1.4$, $b = 0.3$ one observes the so-called Hénon attractor by iterating $h_{a,b}^{n}, n \to \infty$:

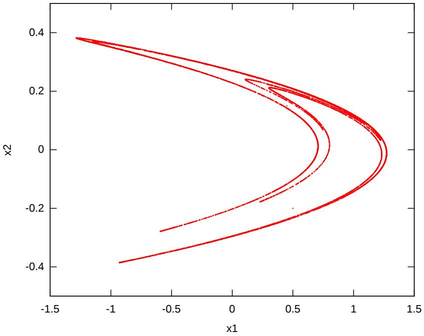


Hénon attractor

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - a x_1^2, b x_1)$$

- Map iterations: $h_{a,b}^{i+1} := h_{a,b} \circ h_{a,b}^{i}, i \in \mathbb{N}^*$.
- For classical parameter values $a = 1.4$, $b = 0.3$ one observes the so-called Hénon attractor by iterating $h_{a,b}^{n}, n \to \infty$:

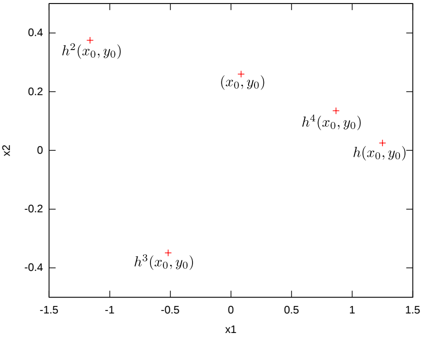


Open question: Is this a Strange Attractor?

Some basic terminology

Orbit of x: $\Gamma(x) = \Gamma^+(x) \cup \Gamma^-(x)$

- Forward orbit:
$$\Gamma^+(x) := \{h^n_{a,b}(x), n \in \mathbb{N}\}$$
- Backward orbit:
$$\Gamma^-(x) := \{y : \exists n \in \mathbb{N} : h^n_{a,b}(y) = x\}$$



Some basic terminology

Orbit of x: $\Gamma(x) = \Gamma^+(x) \cup \Gamma^-(x)$

- Forward orbit:
$$\Gamma^+(x) := \{h^n_{a,b}(x), n \in \mathbb{N}\}$$
- Backward orbit:
$$\Gamma^-(x) := \{y : \exists n \in \mathbb{N} : h^n_{a,b}(y) = x\}$$

Periodic orbit $\Gamma_n(x)$ of period n if $\exists n \in \mathbb{N}^*$ s.t. $h^n_{a,b}(x) = x$.

Some basic terminology

Orbit of x: $\Gamma(x) = \Gamma^+(x) \cup \Gamma^-(x)$

- Forward orbit:
$$\Gamma^+(x) := \{h^n_{a,b}(x), n \in \mathbb{N}\}$$
- Backward orbit:
$$\Gamma^-(x) := \{y : \exists n \in \mathbb{N} : h^n_{a,b}(y) = x\}$$

Periodic orbit $\Gamma_n(x)$ of period n if $\exists n \in \mathbb{N}^*$ s.t. $h^n_{a,b}(x) = x$.

Stable orbit $\Gamma(x)$

Let $d(x, y) = ||x - y||$ and $d(y, \Gamma(x)) = \inf_{z \in \Gamma(x)} d(y, z)$.

$\Gamma(x)$ is stable if given $\varepsilon > 0, \exists \delta > 0$ s.t. $d(h^n_{a,b}(y), \Gamma(x)) < \varepsilon, \forall n \in \mathbb{N}^*$ and $\forall y$ s.t. $d(y, \Gamma(x)) < \delta$.

Some basic terminology

Orbit of x: $\Gamma(x) = \Gamma^+(x) \cup \Gamma^-(x)$

- Forward orbit:
$$\Gamma^+(x) := \{h_{a,b}^n(x), n \in \mathbb{N}\}$$
- Backward orbit:
$$\Gamma^-(x) := \{y : \exists n \in \mathbb{N} : h_{a,b}^n(y) = x\}$$

Periodic orbit $\Gamma_n(x)$ of period n if $\exists n \in \mathbb{N}^*$ s.t. $h_{a,b}^n(x) = x$.

Stable orbit $\Gamma(x)$

Let $d(x,y) = ||x-y||$ and $d(y,\Gamma(x)) = \inf_{z\in\Gamma(x)} d(y,z)$.
$\Gamma(x)$ is stable if given $\varepsilon > 0, \exists\delta > 0$ s.t. $d(h_{a,b}^n(y),\Gamma(x)) < \varepsilon$, $\forall n \in \mathbb{N}^*$ and $\forall y$ s.t. $d(y,\Gamma(x)) < \delta$.

Asymptotically Stable orbit $\Gamma(x)$ (sink)

$\Gamma(x)$ is asymptotically stable if it is stable and (by choosing δ smaller if necessary) $d(h_{a,b}^n(y),\Gamma(x)) \to 0$ as $n \to \infty$.

Some basic terminology II

Attractor

- describes asymptotic behaviour of typical orbits.

Some basic terminology II

Attractor

- describes asymptotic behaviour of typical orbits.
- Attracting Sets
  Let T be a compact set such that $h(T) = T$.
  T is called *attracting* if there exists an open neighborhood U of T such that
  $\bigcap_{i=0}^{\infty} h^i(U) = T$.

Some basic terminology II

Attractor

- describes asymptotic behaviour of typical orbits.

- Attracting Sets
  Let T be a compact set such that $h(T) = T$.
  T is called *attracting* if there exists an open neighborhood U of T such that $\bigcap_{i=0}^{\infty} h^i(U) = T$.

- Attractor: An attracting set which contains a dense orbit.

Some basic terminology II

Attractor

- describes asymptotic behaviour of typical orbits.

- Attracting Sets
  Let T be a compact set such that $h(T) = T$.
  T is called *attracting* if there exists an open neighborhood U of T such that
  $\bigcap_{i=0}^{\infty} h^i(U) = T$.

- Attractor: An attracting set which contains a dense orbit.
- Basin of attraction, $B(T)$: largest such U.

Some basic terminology II

Attractor

- describes asymptotic behaviour of typical orbits.

- Attracting Sets
  Let T be a compact set such that $h(T) = T$.
  T is called *attracting* if there exists an open neighborhood U of T such that
  $\bigcap_{i=0}^{\infty} h^i(U) = T$.

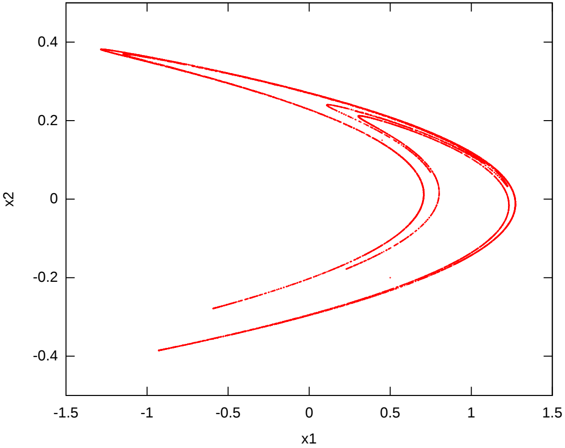
- Attractor: An attracting set which contains a dense orbit.
- Basin of attraction, $B(T)$: largest such U.

Strange Attractor

For *almost all* pairs of points $x, y \in B(T)$ there exists $k \in \mathbb{N}^*$ s.t. $h^k(x)$ and $h^k(y)$ separate by at least a constant δ_h.

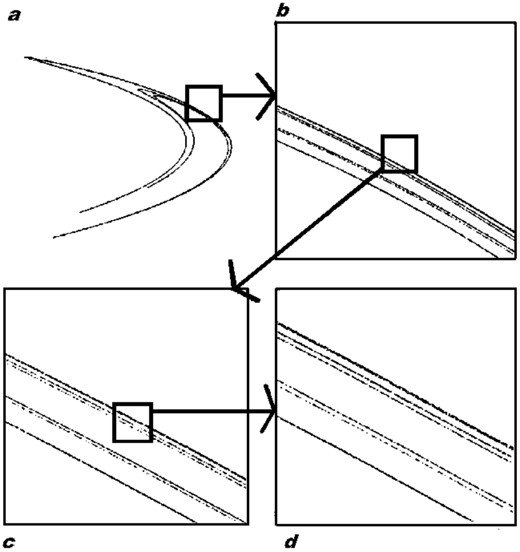
Hénon Map

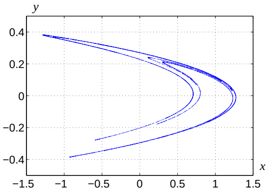
$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$



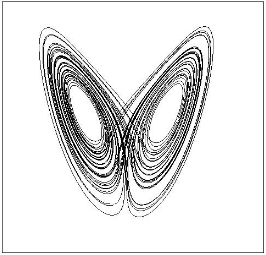
Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

Hénon Attractor - Fractal Dimension





(a) Hénon Attractor



(b) Lorenz Attractor

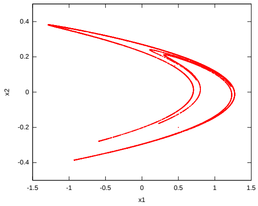
(c) Ueda Attractor

- Name coined by Takens and Ruelle $\simeq 1971$

Ruelle (The Mathematical Intelligencer 2, 126, 1980): The name is beautiful, and well suited to these astonishing objects, of which we understand so little.

Hénon Map

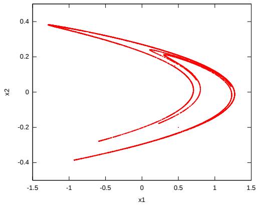
$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$



Open Question: Is Hénon Attractor a Strange Attractor?

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

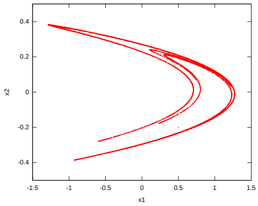


- Chaotic map: aperiodic trajectories (generally believed)

Open Question: Is Hénon Attractor a Strange Attractor?

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

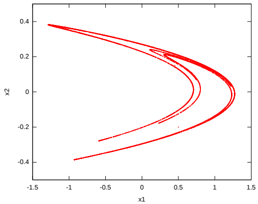


- Chaotic map: aperiodic trajectories (generally believed)
- There is a set of parameters (near $b = 0$) with positive Lebesgue measure for which the Hénon map has a strange attractor. [Benedicks & Carleson,'91].

Open Question: Is Hénon Attractor a Strange Attractor?

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$



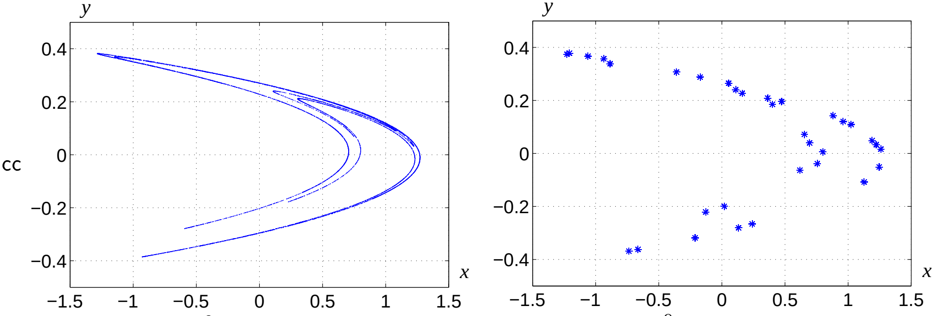
- Chaotic map: aperiodic trajectories (generally believed)
- There is a set of parameters (near $b = 0$) with positive Lebesgue measure for which the Hénon map has a strange attractor. [Benedicks & Carleson,'91].
- The parameters space is believed to be densely filled with regions, where the attractor is periodic.

Example of sink in Hénon attractor [Galias & Tucker 2013]

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - a x_1^2, b x_1)$$

Let $a = 1.399999486944$, $b = 0.3$. Trajectory composed of 10000 points:



(a) $5 \cdot 10^9$ iterations skipped

(b) $6 \cdot 10^9$ iterations skipped.

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Hénon attractor

Hénon Map

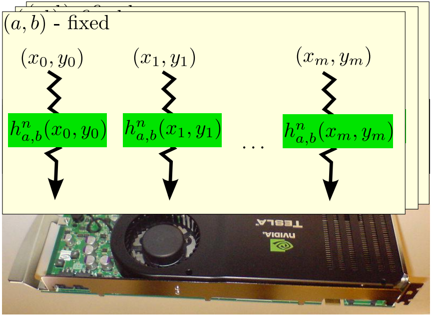
$$h_{a,b}(x_1, x_2) = (1 + x_2 - a x_1^2, b x_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6, \ldots, 10^9$.

High parallelism → Graphics Processing Units (GPUs)



Hénon Map

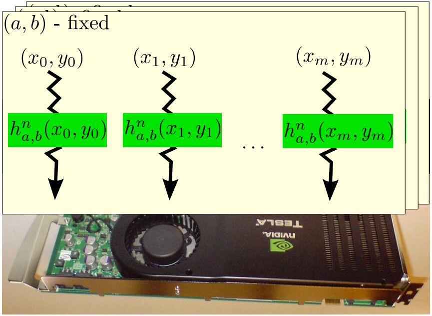
$$h_{a,b}(x_1, x_2) = (1 + x_2 - a x_1^2, b x_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6, \dots, 10^9$.
   - chaotic $\rightarrow$ multiple precision

High parallelism $\rightarrow$ Graphics Processing Units (GPUs)



Hénon attractor

Hénon Map

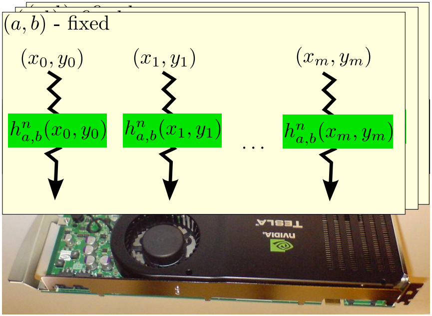
$$h_{a,b}(x_1, x_2) = (1 + x_2 - a x_1^2, b x_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6, \ldots, 10^9$.
   - chaotic $\rightarrow$ multiple precision

High parallelism $\rightarrow$ Graphics Processing Units (GPUs)



Hénon attractor

Hénon Map

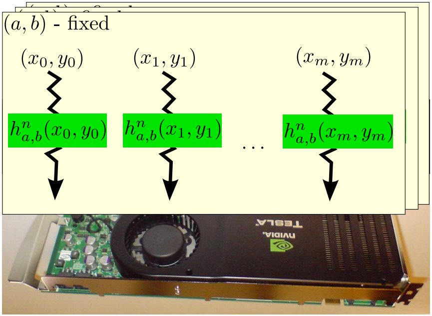
$$h_{a,b}(x_1, x_2) = (1 + x_2 - a x_1^2, b x_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6, \ldots, 10^9$.
   - chaotic $\rightarrow$ multiple precision
2. A posteriori validation of existence and stability with interval arithmetic

High parallelism $\rightarrow$ Graphics Processing Units (GPUs)



Hénon attractor

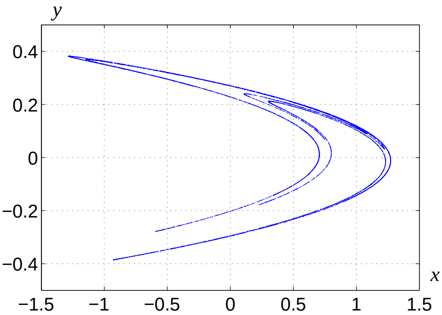
Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6, \dots, 10^9$.



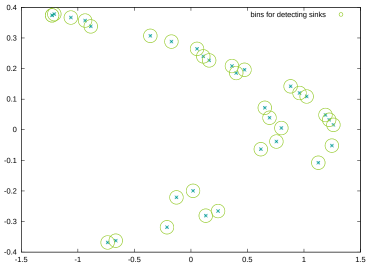
Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - a x_1^2, b x_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6, \ldots, 10^9$.
   - chaotic $\rightarrow$ multiple precision



Hénon attractor

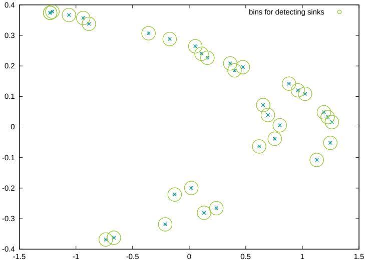
Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - a x_1^2, b x_1)$$

Goal: Given (a, b), prove existence of sinks (stable periodic orbits).

Method:

1. Find numerical approximation of sinks
   - Typical values: 10^6 parameters, 10^3 initial values, orbit length $10^6, \ldots, 10^9$.
   - chaotic $\rightarrow$ multiple precision
   - Detected no. of bins estimate period length



Theorem

Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $F \in \mathcal{C}^1(D)$, $\boldsymbol{x} \in \mathbb{ID}$, $\hat{x} \in \boldsymbol{x}$.

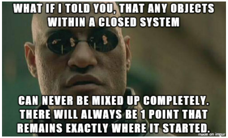
$$N(\boldsymbol{x}) := \hat{x} - F'(\boldsymbol{x})^{-1} F(\hat{x})$$

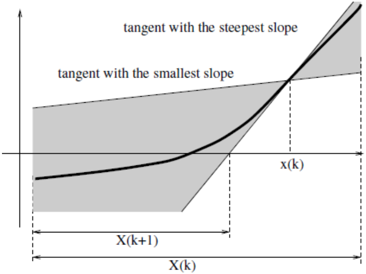
If $N(\boldsymbol{x})$ is well-defined, then the following statements hold:

(1) if $\boldsymbol{x}$ contains a zero x^* of F, then so does $N(\boldsymbol{x}) \cap \boldsymbol{x}$;

(2) if $N(\boldsymbol{x}) \cap \boldsymbol{x} = \emptyset$, then $\boldsymbol{x}$ has no zeros of F;

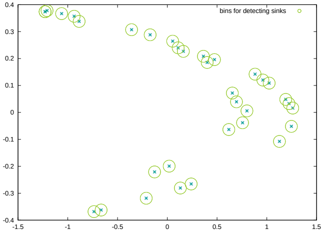
(3) if $N(\boldsymbol{x}) \subseteq \boldsymbol{x}$, then $\boldsymbol{x}$ contains a unique zero of F;

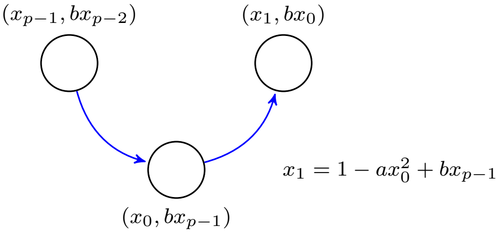




$\mathbb{ID}$ is the set of all intervals included in D.

Validation of periodic orbits using Newton operator [Galias 2001], [Galias & Tucker, 2011, 2013]



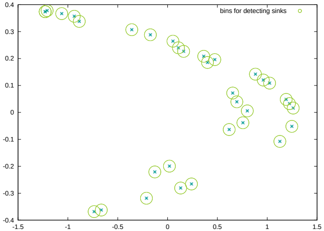


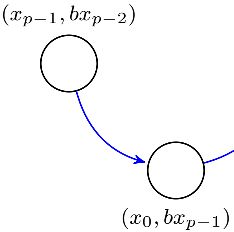
- Let $F : \mathbb{R}^p \to \mathbb{R}^p$ defined by:

$$[F(x)]_k = 1 - ax_k^2 + bx_{(k-1) \bmod p} - x_{(k+1) \bmod p}, \quad k = 0, \ldots, p-1.$$

- Then $F(x_0, \ldots, x_p) = 0$ iif $z_0 = (x_0, y_0) = (x_0, bx_{p-1})$ is a fixed point of h^p

Validation of periodic orbits using Newton operator [Galias 2001], [Galias & Tucker, 2011, 2013]





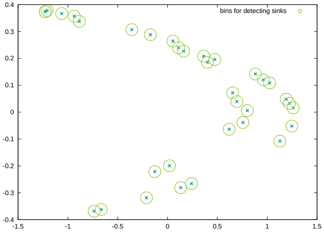
$$x_1 = 1 - ax_0^2 + bx_{p-1}$$

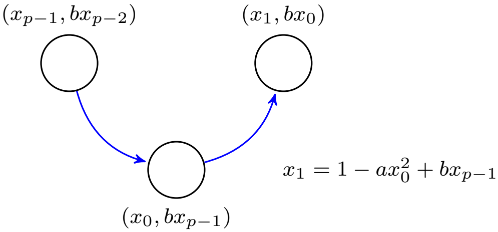
- Let $F : \mathbb{R}^p \to \mathbb{R}^p$ defined by:

$$[F(x)]_k = 1 - ax_k^2 + bx_{(k-1) \bmod p} - x_{(k+1) \bmod p}, \quad k = 0, \ldots, p-1.$$

- Then $F(x_0, \ldots, x_p) = 0$ iif $z_0 = (x_0, y_0) = (x_0, bx_{p-1})$ is a fixed point of h^p
- Choose $r > 0$ s.t. $\boldsymbol{x}_k = [\hat{x}_k - r, \hat{x}_k + r]$.

Validation of periodic orbits using Newton operator [Galias 2001], [Galias & Tucker, 2011, 2013]





- Let $F : \mathbb{R}^p \to \mathbb{R}^p$ defined by:

$$[F(x)]_k = 1 - ax_k^2 + bx_{(k-1) \bmod p} - x_{(k+1) \bmod p}, \quad k = 0, \ldots, p-1.$$

- Then $F(x_0, \ldots, x_p) = 0$ iif $z_0 = (x_0, y_0) = (x_0, bx_{p-1})$ is a fixed point of h^p

- Choose $r > 0$ s.t. $\boldsymbol{x}_k = [\hat{x}_k - r, \hat{x}_k + r]$.
- Verify that $N(\boldsymbol{x}) \subseteq \boldsymbol{x}$, with:

$$N(\boldsymbol{x}) := \hat{x} - F'(\boldsymbol{x})^{-1} F(\hat{x})$$

Validation of periodic orbits using Newton operator [Galias 2001], [Galias & Tucker, 2011, 2013]

- Let $F : \mathbb{R}^p \to \mathbb{R}^p$ defined by:

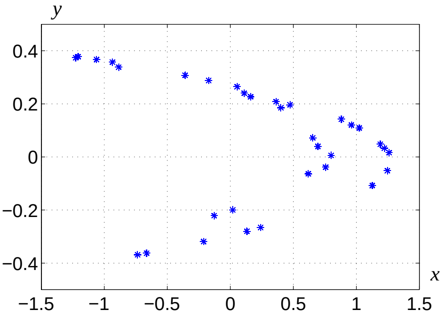
$$[F(x)]_k = 1 - ax_k^2 + bx_{(k-1) \bmod p} - x_{(k+1) \bmod p}, \quad k = 0, \dots, p-1.$$

- Numerical approx of sink $\hat{x}$ given.
- Choose $r > 0$ s.t. $\boldsymbol{x}_k = [\hat{x}_k - r, \hat{x}_k + r]$.
- Verify that $N(\boldsymbol{x}) \subseteq \boldsymbol{x}$, with:

$$N(\boldsymbol{x}) := \hat{x} - \underbrace{F'(\boldsymbol{x})^{-1} F(\hat{x})}_{\boldsymbol{y}}$$

$$\begin{bmatrix} -2a\boldsymbol{x_0} & -1 & 0 & \cdots & b \\ b & -2a\boldsymbol{x_1} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & \cdots & b & -2a\boldsymbol{x_{p-2}} & -1 \\ -1 & 0 & \cdots & b & -2a\boldsymbol{x_{p-1}} \end{bmatrix} \cdot \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{p-2} \\ y_{p-1} \end{bmatrix} = \begin{bmatrix} F(\hat{x})_0 \\ F(\hat{x})_1 \\ \vdots \\ F(\hat{x})_{p-2} \\ F(\hat{x})_{p-1} \end{bmatrix}$$

Let $a = 1.399999486944$, $b = 0.3$.



Validation example: $\hat{x}$=`-1.22783854559, -0.73659038778,..., 1.246771101387` and $r = 10^{-7}$.

Examples of found Hénon sinks [Galias & Tucker, 2013]

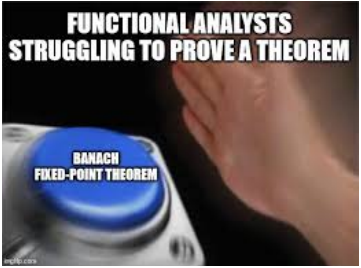
a	p	w	b_r	λ_1
1.399922051	25	$5.522{-}12$	$2.473{-}12$	-0.00132
1.39997174948	30	$1.354{-}11$	$3.561{-}12$	-0.01887
1.3999769102	18	$3.207{-}09$	$1.014{-}09$	-0.05306
1.39998083519	24	$1.703{-}11$	$7.384{-}12$	-0.02819
1.399984477	20	$8.875{-}10$	$4.076{-}10$	-0.05099
1.39999492185	22	$3.686{-}11$	$1.531{-}11$	-0.09600
1.3999964733062	39	$2.784{-}13$	$1.115{-}13$	-0.03547
1.399999486944	33	$1.110{-}12$	$6.901{-}13$	-0.01843
1.40000929916	25	$1.118{-}11$	$5.128{-}12$	-0.08379
1.4000227433	21	$2.262{-}10$	$7.901{-}11$	-0.05612
1.40002931695	27	$5.782{-}11$	$2.646{-}11$	-0.01140
1.40006377472	27	$8.692{-}11$	$3.810{-}11$	-0.05636
1.40006667358	24	$6.278{-}11$	$2.646{-}11$	-0.01112
1.4000843045	27	$9.400{-}11$	$4.572{-}11$	-0.06870
1.40009110518	22	$3.493{-}11$	$1.531{-}11$	-0.02157
1.4000967515	26	$2.463{-}10$	$1.365{-}10$	-0.13233

Note: The orbit stability depends on the eigenvalues of $J_p(z_0)$ being inside the unit circle, with,

$$J_p(z_0) = (h^p)'(z_0) = h'(h^{p-1}(z_0)) \cdots h'(h(z_0)) \cdot h'(z_0).$$

Part 2: Validated approximation of solutions of Newton-like operators

Part 2: Validated approximation of solutions of Newton-like operators



Compute Bounds with Contraction Mapping principle

Banach Fixed Point Theorem

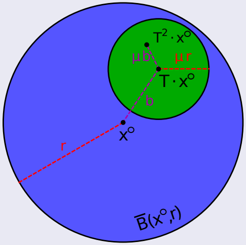
(X, d) a complete metric space, $\mathbf{T} : X \to X$, $x^\circ \in X$, and compute μ, b, r s.t.

- $d(x^\circ, \mathbf{T} \cdot x^\circ) \ \leq \ b$
- $\mathbf{T}$ is μ-Lipschitz over the closed ball $\bar{B}(x^\circ, r) := \{x \in X \mid d(x, x^\circ) \leq r\}$:

$$d(\mathbf{T} \cdot x_1, \mathbf{T} \cdot x_2) \ \leq \ \mu d(x_1, x_2), \qquad x_1, x_2 \in \bar{B}(x^\circ, r)$$

Also ensure that:

- $\mu < 1$ — $\mathbf{T}$ is contracting over $\bar{B}(x^\circ, r)$
- $b + \mu r \ \leq \ r$ — $\bar{B}(x^\circ, r)$ is strongly stable



Then $\mathbf{T}$ admits a unique fixed-point x^* in $\bar{B}(x^\circ, r)$.

A Posteriori Newton-like Validation Methods

Target: $x^* \in X$ solution of $\mathbf{F} \; \cdot x = 0$ where $\mathbf{F} : X \to Y$ is differentiable

A Posteriori Newton-like Validation Methods

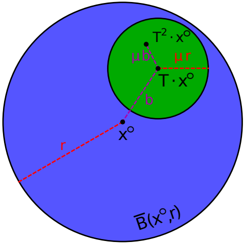
Target: $x^* \in X$ solution of $\mathbf{F} \;\cdot x = 0$ where $\mathbf{F} : X \to Y$ is differentiable

- Compute a candidate approximation $x^\circ \in X$

Target: $x^* \in X$ solution of $\mathbf{F} \cdot x = 0$ where $\mathbf{F} : X \to Y$ is differentiable

- Compute a candidate approximation $x^\circ \in X$
- Newton-like operator $\mathbf{T} : X \to X, \quad x \mapsto x - \mathbf{A} \cdot \mathbf{F} \cdot x$:

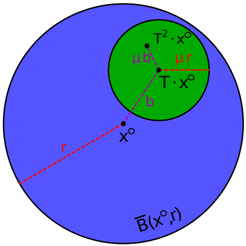
$$\text{with} \quad \mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$$



Target: $x^* \in X$ solution of $\mathbf{F} \cdot x = 0$ where $\mathbf{F} : X \to Y$ is differentiable

- Compute a candidate approximation $x^\circ \in X$
- Newton-like operator $\mathbf{T} : X \to X, \quad x \mapsto x - \mathbf{A} \cdot \mathbf{F} \cdot x$:

with $\mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$ $\qquad b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$



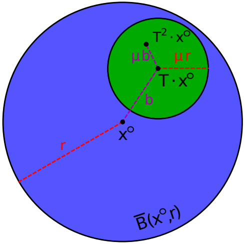
Target: $x^* \in X$ solution of $\mathbf{F} \cdot x = 0$ where $\mathbf{F} : X \to Y$ is differentiable

- Compute a candidate approximation $x^\circ \in X$
- Newton-like operator $\mathbf{T} : X \to X, \quad x \mapsto x - \mathbf{A} \cdot \mathbf{F} \cdot x$:

with $\mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$ $\qquad b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$

- Compute a rigorous Lipschitz constant for $\mathbf{T}$ over $\bar{B}(x^\circ, r)$:

$$\mu \geq \sup_{x \in \bar{B}(x^\circ, r)} \|\mathrm{D}\mathbf{T}_x\| = \sup_{x \in \bar{B}(x^\circ, r)} \|\mathbf{1}_X - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_x\|$$



Target: $x^* \in X$ solution of $\mathbf{F} \cdot x = 0$ where $\mathbf{F} : X \to Y$ is differentiable

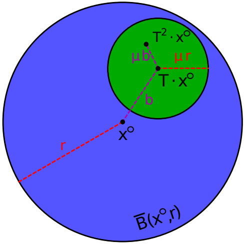
- Compute a candidate approximation $x^\circ \in X$
- Newton-like operator $\mathbf{T} : X \to X, \quad x \mapsto x - \mathbf{A} \cdot \mathbf{F} \cdot x$:

with $\mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$ $\qquad b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$

- Compute a rigorous Lipschitz constant for $\mathbf{T}$ over $\bar{B}(x^\circ, r)$:

$$\mu \geq \sup_{x \in \bar{B}(x^\circ, r)} \|\mathrm{D}\mathbf{T}_x\| = \sup_{x \in \bar{B}(x^\circ, r)} \|\mathbf{1}_X - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_x\|$$

  - μ does not depend on r if $\mathbf{F}$ is affine
    $\Rightarrow$ Choose $r := b/(1-\mu)$



A Posteriori Newton-like Validation Methods

Target: $x^* \in X$ solution of $\mathbf{F} \cdot x = 0$ where $\mathbf{F} : X \to Y$ is differentiable

- Compute a candidate approximation $x^\circ \in X$
- Newton-like operator $\mathbf{T} : X \to X, \quad x \mapsto x - \mathbf{A} \cdot \mathbf{F} \cdot x$:

$$\text{with} \quad \mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1} \qquad\qquad b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$$

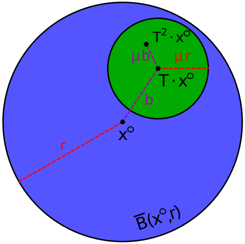
- Compute a rigorous Lipschitz constant for $\mathbf{T}$ over $\bar{B}(x^\circ, r)$:

$$\mu \geq \sup_{x \in \bar{B}(x^\circ, r)} \|\mathrm{D}\mathbf{T}_x\| = \sup_{x \in \bar{B}(x^\circ, r)} \|\mathbf{1}_X - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_x\|$$

  - μ does not depend on r if $\mathbf{F}$ is affine
    $\Rightarrow$ Choose $r := b/(1-\mu)$
  - Otherwise, r and $\mu(r)$ must satisfy:

$$b + \mu(r) r \leq r$$

    which usually expands to a polynomial equation over r



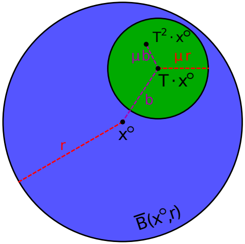
Target: $x^* \in X$ solution of $\mathbf{F} \cdot x = 0$ where $\mathbf{F} : X \to Y$ is differentiable

- Compute a candidate approximation $x^\circ \in X$
- Newton-like operator $\mathbf{T} : X \to X, \quad x \mapsto x - \mathbf{A} \cdot \mathbf{F} \cdot x$:

$$\text{with} \quad \mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1} \qquad\qquad b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$$

- Compute a rigorous Lipschitz constant for $\mathbf{T}$ over $\bar{B}(x^\circ, r)$:

$$\mu \geq \sup_{x \in \bar{B}(x^\circ, r)} \|\mathrm{D}\mathbf{T}_x\| = \sup_{x \in \bar{B}(x^\circ, r)} \|\mathbf{1}_X - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_x\|$$



  - μ does not depend on r if $\mathbf{F}$ is affine
    $\Rightarrow$ Choose $r := b/(1 - \mu)$
  - Otherwise, r and $\mu(r)$ must satisfy:

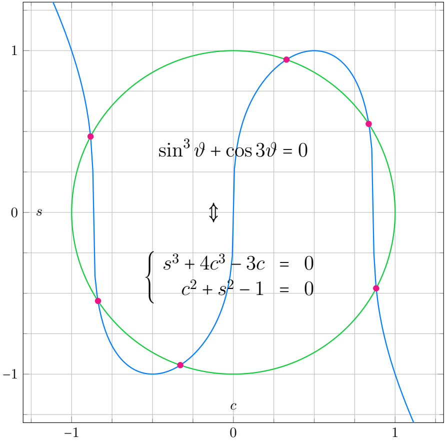
$$b + \mu(r) r \leq r$$

which usually expands to a polynomial equation over r

$\Rightarrow$ $\mathbf{F}$ has a unique root x^* in $\bar{B}(x^\circ, r)$.

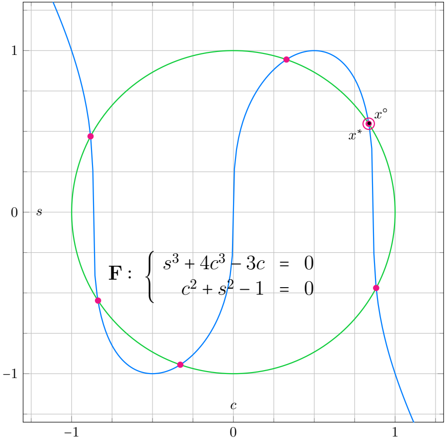
A Posteriori Newton-like Validation Methods

- Goal: Rigorously approximate $x^* \in X$, solution of $\mathbf{F} \cdot x = 0$ with $\mathbf{F} : X \to Y$.



Example courtesy of F. Bréhard, *Certified Numerics in Function Spaces*. PhD Thesis, 2019.

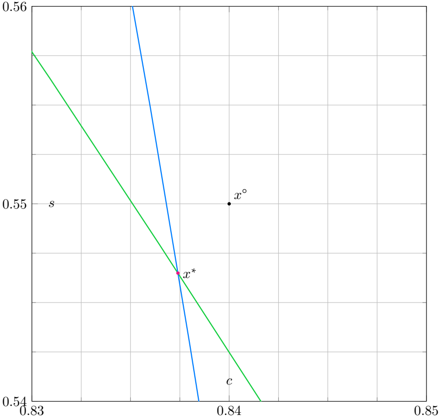
- Goal: Rigorously approximate $x^* \in X$, solution of $\mathbf{F} \cdot x = 0$ with $\mathbf{F} : X \to Y$.



- Compute an approx $x^\circ \in X$

Example courtesy of F. Bréhard, *Certified Numerics in Function Spaces*. PhD Thesis, 2019.

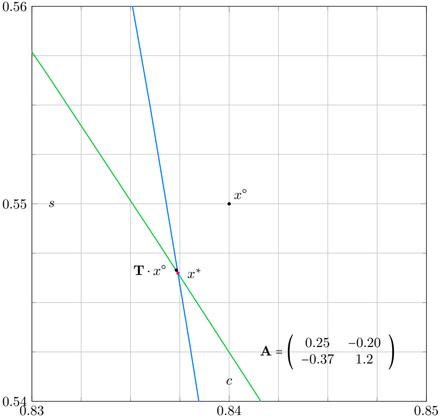
- Goal: Rigorously approximate $x^* \in X$, solution of $\mathbf{F} \cdot x = 0$ with $\mathbf{F} : X \to Y$.



  - Compute an approx $x^\circ \in X$

Example courtesy of F. Bréhard, *Certified Numerics in Function Spaces*. PhD Thesis, 2019.

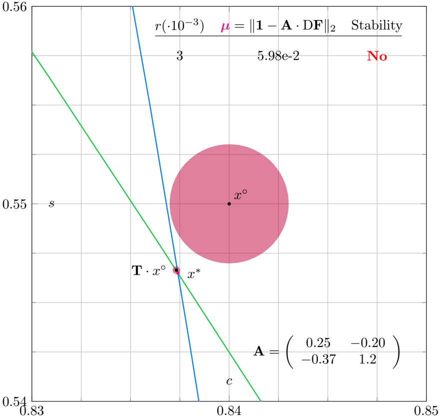
- **Goal**: Rigorously approximate $x^* \in X$, solution of $\mathbf{F} \cdot x = 0$ with $\mathbf{F} : X \to Y$.



- Compute an approx $x^\circ \in X$
- Build Newton-like operator
$\mathbf{T} \cdot x = x - \mathbf{A} \cdot \mathbf{F} \cdot x, \quad \mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$
- Bound $b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$

Example courtesy of F. Bréhard, *Certified Numerics in Function Spaces*. PhD Thesis, 2019.

- **Goal**: Rigorously approximate $x^* \in X$, solution of $\mathbf{F} \cdot x = 0$ with $\mathbf{F} : X \to Y$.



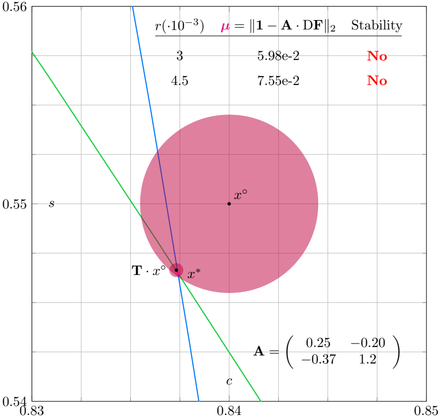
- Compute an approx $x^\circ \in X$
- Build Newton-like operator
  $\mathbf{T} \cdot x = x - \mathbf{A} \cdot \mathbf{F} \cdot x, \quad \mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$
- Bound $b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$
- Bound Lipschitz ratio over $\bar{B}(x^\circ, r)$:

$$\mu(r) \geq \sup_{x \in \bar{B}(x^\circ, r)} \|\mathrm{D}\mathbf{T}_x\| = \sup_{x \in \bar{B}(x^\circ, r)} \|\mathbf{1}_X - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_x\|$$

- Choose r such that $\quad b + \mu(r) r \leq r$

Example courtesy of F. Bréhard, *Certified Numerics in Function Spaces*. PhD Thesis, 2019.

- Goal: Rigorously approximate $x^* \in X$, solution of $\mathbf{F} \cdot x = 0$ with $\mathbf{F} : X \to Y$.



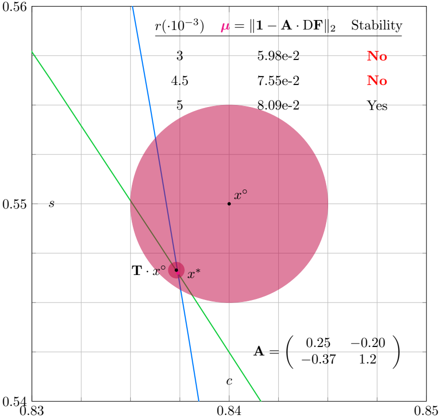
- Compute an approx $x^\circ \in X$
- Build Newton-like operator
  $\mathbf{T} \cdot x = x - \mathbf{A} \cdot \mathbf{F} \cdot x, \quad \mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$
- Bound $b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$
- Bound Lipschitz ratio over $\bar{B}(x^\circ, r)$:

$$\mu(r) \geq \sup_{x \in \bar{B}(x^\circ, r)} \|\mathrm{D}\mathbf{T}_x\| = \sup_{x \in \bar{B}(x^\circ, r)} \|\mathbf{1}_X - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_x\|$$

- Choose r such that $\quad b + \mu(r) r \leq r$

Example courtesy of F. Bréhard, *Certified Numerics in Function Spaces*. PhD Thesis, 2019.

- Goal: Rigorously approximate $x^* \in X$, solution of $\mathbf{F} \cdot x = 0$ with $\mathbf{F} : X \to Y$.



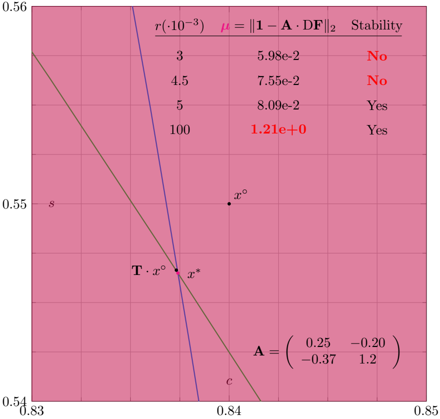
- Compute an approx $x^\circ \in X$
- Build Newton-like operator
  $\mathbf{T} \cdot x = x - \mathbf{A} \cdot \mathbf{F} \cdot x, \quad \mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$
- Bound $b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$
- Bound Lipschitz ratio over $\bar{B}(x^\circ, r)$:

$$\mu(r) \geq \sup_{x \in \bar{B}(x^\circ, r)} \|\mathrm{D}\mathbf{T}_x\| = \sup_{x \in \bar{B}(x^\circ, r)} \|\mathbf{1}_X - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_x\|$$

- Choose r such that $\quad b + \mu(r) r \leq r$

Example courtesy of F. Bréhard, *Certified Numerics in Function Spaces*. PhD Thesis, 2019.

- **Goal**: Rigorously approximate $x^* \in X$, solution of $\mathbf{F} \cdot x = 0$ with $\mathbf{F} : X \to Y$.



$r(\cdot 10^{-3})$	$\mu = \|\mathbf{1} - \mathbf{A} \cdot \mathrm{D}\mathbf{F}\|_2$	Stability
3	5.98e-2	**No**
4.5	7.55e-2	**No**
5	8.09e-2	Yes
100	**1.21e+0**	Yes

- Compute an approx $x^\circ \in X$
- Build Newton-like operator
  $\mathbf{T} \cdot x = x - \mathbf{A} \cdot \mathbf{F} \cdot x, \quad \mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$
- Bound $b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$
- Bound Lipschitz ratio over $\bar{B}(x^\circ, r)$:

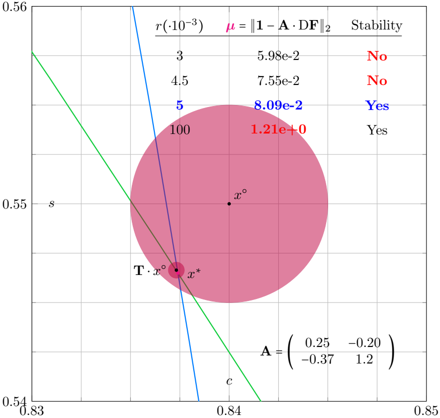
$$\mu(r) \geq \sup_{x \in \bar{B}(x^\circ, r)} \|\mathrm{D}\mathbf{T}_x\| = \sup_{x \in \bar{B}(x^\circ, r)} \|\mathbf{1}_X - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_x\|$$

- Choose r such that $\quad b + \mu(r) r \leq r$

Example courtesy of F. Bréhard, *Certified Numerics in Function Spaces*. PhD Thesis, 2019.

A Posteriori Newton-like Validation Methods

▶ Goal: Rigorously approximate $x^* \in X$, solution of $\mathbf{F} \cdot x = 0$ with $\mathbf{F} : X \to Y$.



- Compute an approx $x^\circ \in X$
- Build Newton-like operator
  $\mathbf{T} \cdot x = x - \mathbf{A} \cdot \mathbf{F} \cdot x, \quad \mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$
- Bound $b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$
- Bound Lipschitz ratio over $\bar{B}(x^\circ, r)$:

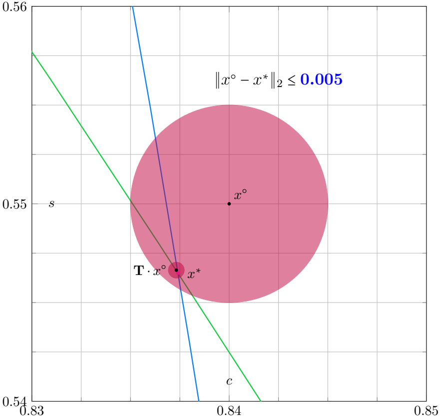
$$\mu(r) \geq \sup_{x \in \bar{B}(x^\circ, r)} \|\mathrm{D}\mathbf{T}_x\| = \sup_{x \in \bar{B}(x^\circ, r)} \|\mathbf{1}_X - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_x\|$$

- Choose r such that $\quad b + \mu(r) r \leq r$

Example courtesy of F. Bréhard, *Certified Numerics in Function Spaces*. PhD Thesis, 2019.

A Posteriori Newton-like Validation Methods

- Goal: Rigorously approximate $x^* \in X$, solution of $\mathbf{F} \cdot x = 0$ with $\mathbf{F} : X \to Y$.



- Compute an approx $x^\circ \in X$
- Build Newton-like operator
$\mathbf{T} \cdot x = x - \mathbf{A} \cdot \mathbf{F} \cdot x, \quad \mathbf{A} \approx (\mathrm{D}\mathbf{F}_{x^\circ})^{-1}$
- Bound $b := \|\mathbf{T} \cdot x^\circ - x^\circ\|$
- Bound Lipschitz ratio over $\bar{B}(x^\circ, r)$:

$$\mu(r) \geq \sup_{x \in \bar{B}(x^\circ, r)} \|\mathrm{D}\mathbf{T}_x\| = \sup_{x \in \bar{B}(x^\circ, r)} \|\mathbf{1}_X - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_x\|$$

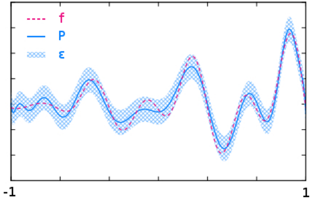
- Choose r such that $\quad b + \mu(r) r \leq r$

Banach Fixed-Point Theorem

$\mathbf{F}$ has a unique root x^* in $\bar{B}(x^\circ, r)$

Example courtesy of F. Bréhard, *Certified Numerics in Function Spaces*. PhD Thesis, 2019.

Validated approximations for the reciprocal and square-root of a function
(with applications to Computer-Assisted Proofs
and Hilbert's 16th Problem)



Definition

A pair $(P, \varepsilon) \in \mathbb{R}[X] \times \mathbb{R}_+$ is a rigorous polynomial approximation (RPA) of f for a given norm $\|\cdot\|$ if $\|f - P\| \leq \varepsilon$.

Example: sup-norm over $[-1, 1]$:

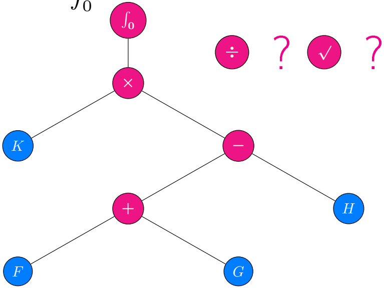
$$f \in (P, \varepsilon) \Leftrightarrow |f(t) - P(t)| \leq \varepsilon \quad \forall t \in [-1, 1]$$

Some elementary operations:

- $(P, \varepsilon) + (Q, \eta) := (P + Q, \varepsilon + \eta)$,
- $(P, \varepsilon) - (Q, \eta) := (P - Q, \varepsilon + \eta)$,
- $(P, \varepsilon) \cdot (Q, \eta) := (PQ, \|Q\|\eta + \|P\|\varepsilon + \eta\varepsilon)$
- $\int_0 (P, \varepsilon) := (\int_0^t P(s)\mathrm{d}s, \varepsilon)$

Example:

$$r(t) = \int_0^t k(s)(f(s) + g(s) - h(s))\mathrm{d}s$$



RPAs for the reciprocal or division

Given $f, g \in \mathcal{C}([-1,1])$, compute an approximation $\boldsymbol{x}^\circ \approx \frac{f}{g}$ and an error bound $\left\|\boldsymbol{x}^\circ - \frac{f}{g}\right\|_\infty$.

Strategy:

- Newton-like operator $\mathbf{T}$ with unique fixed point $\boldsymbol{x}^\star = \frac{f}{g}$:

$$\mathbf{T} \cdot x = x - \tilde{\psi}(gx - f) \qquad \tilde{\psi} \approx 1/g$$

RPAs for the reciprocal or division

Given $f, g \in \mathcal{C}([-1,1])$, compute an approximation $\boldsymbol{x}^\circ \approx \frac{f}{g}$ and an error bound $\left\| \boldsymbol{x}^\circ - \frac{f}{g} \right\|_\infty$.

Strategy:

- Newton-like operator $\mathbf{T}$ with unique fixed point $\boldsymbol{x}^\star = \frac{f}{g}$:

$$\mathbf{T} \cdot x = x - \tilde{\psi}(gx - f) \qquad \tilde{\psi} \approx 1/g$$

- Is $\mathbf{T}$ contracting?

$$\|\mathrm{D}\mathbf{T}\| = \|1 - \tilde{\psi} g\|_\infty = \mu < 1$$

RPAs for the reciprocal or division

Given $f, g \in \mathcal{C}([-1,1])$, compute an approximation $\boldsymbol{x}^\circ \approx \frac{f}{g}$ and an error bound $\left\|\boldsymbol{x}^\circ - \frac{f}{g}\right\|_\infty$.

Strategy:

- Newton-like operator $\mathbf{T}$ with unique fixed point $\boldsymbol{x}^\star = \frac{f}{g}$:

$$\mathbf{T} \cdot x = x - \tilde{\psi}(gx - f) \qquad \tilde{\psi} \approx 1/g$$

- Is $\mathbf{T}$ contracting?

$$\|\mathrm{D}\mathbf{T}\| = \|1 - \tilde{\psi} g\|_\infty = \mu < 1$$

- Apply the Banach fixed-point theorem:

$$\|\boldsymbol{x}^\circ - \mathbf{T} \cdot \boldsymbol{x}^\circ\|_\infty = \|\tilde{\psi}(g\boldsymbol{x}^\circ - f)\|_\infty \leq b$$

Given $f, g \in \mathcal{C}([-1,1])$, compute an approximation $\boldsymbol{x}^\circ \approx \frac{f}{g}$ and an error bound $\left\| \boldsymbol{x}^\circ - \frac{f}{g} \right\|_\infty$.

Strategy:

- Newton-like operator $\mathbf{T}$ with unique fixed point $\boldsymbol{x}^\star = \frac{f}{g}$:

$$\mathbf{T} \cdot x = x - \tilde{\psi}(gx - f) \qquad \tilde{\psi} \approx 1/g$$

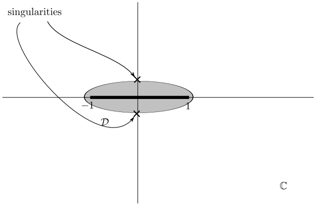
- Is $\mathbf{T}$ contracting?

$$\|\mathrm{D}\mathbf{T}\| = \|1 - \tilde{\psi} g\|_\infty = \mu < 1$$

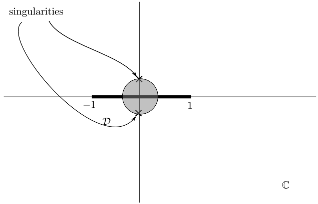
- Apply the Banach fixed-point theorem:

$$\|\boldsymbol{x}^\circ - \mathbf{T} \cdot \boldsymbol{x}^\circ\|_\infty = \|\tilde{\psi}(g\boldsymbol{x}^\circ - f)\|_\infty \leq b \quad \Rightarrow \quad \|\boldsymbol{x}^\circ - \boldsymbol{x}^\star\|_\infty \leq r := b/(1-\mu)$$

RPAs for the reciprocal or division



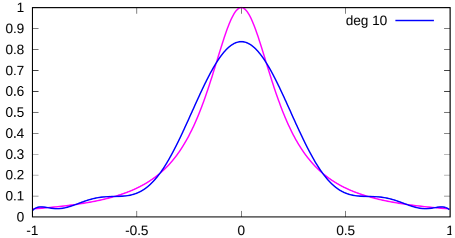
Chebyshev series



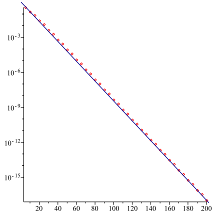
Taylor series

Example

Compute $\boldsymbol{x}^\circ(t) \approx \dfrac{1}{1+\varepsilon t^2}$ and $r \geq \left\| \boldsymbol{x}^\circ(t) - \dfrac{1}{1+\varepsilon t^2} \right\|_\infty$ for $t \in [-1,1]$ and fixed ε.



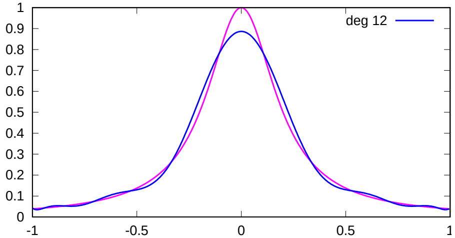
Chebyshev series approx ($\varepsilon = 25$)



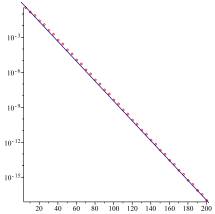
Coeffs' convergence rate

Example

Compute $\boldsymbol{x}^\circ(t) \approx \dfrac{1}{1+\varepsilon t^2}$ and $r \geq \left\| \boldsymbol{x}^\circ(t) - \dfrac{1}{1+\varepsilon t^2} \right\|_\infty$ for $t \in [-1, 1]$ and fixed ε.



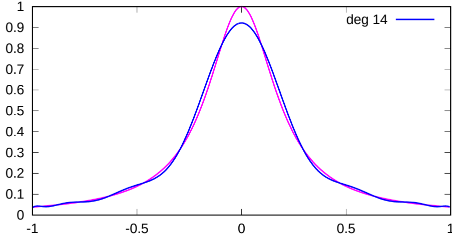
Chebyshev series approx ($\varepsilon = 25$)



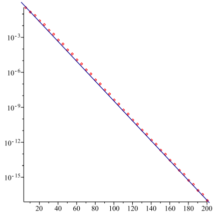
Coeffs' convergence rate

Example

Compute $\boldsymbol{x}^{\circ}(t) \approx \frac{1}{1+\varepsilon t^2}$ and $r \geq \left\| \boldsymbol{x}^{\circ}(t) - \frac{1}{1+\varepsilon t^2} \right\|_{\infty}$ for $t \in [-1, 1]$ and fixed ε.



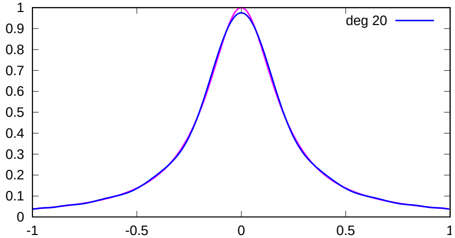
Chebyshev series approx ($\varepsilon = 25$)



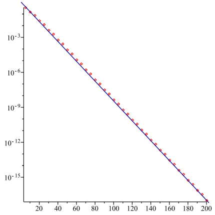
Coeffs' convergence rate

Example

Compute $\boldsymbol{x}^{\circ}(t) \approx \dfrac{1}{1+\varepsilon t^2}$ and $r \geq \left\| \boldsymbol{x}^{\circ}(t) - \dfrac{1}{1+\varepsilon t^2} \right\|_{\infty}$ for $t \in [-1, 1]$ and fixed ε.



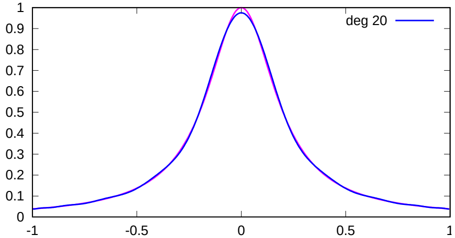
Chebyshev series approx ($\varepsilon = 25$)



Coeffs' convergence rate

Example

Compute $\boldsymbol{x}^{\circ}(t) \approx \dfrac{1}{1+\varepsilon t^2}$ and $r \geq \left\| \boldsymbol{x}^{\circ}(t) - \dfrac{1}{1+\varepsilon t^2} \right\|_{\infty}$ for $t \in [-1, 1]$ and fixed ε.

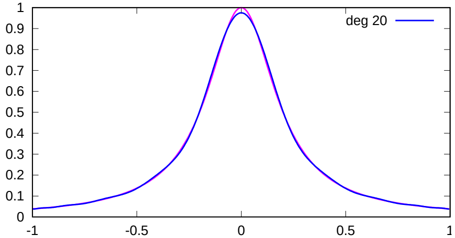


Chebyshev series approx ($\varepsilon = 25$)

deg	$\|\cdot\|_\infty$	
10	0.17	
12	0.12	
14	0.079	
20	0.025	

Example

Compute $\boldsymbol{x}^\circ(t) \approx \dfrac{1}{1+\varepsilon t^2}$ and $r \geq \left\| \boldsymbol{x}^\circ(t) - \dfrac{1}{1+\varepsilon t^2} \right\|_\infty$ for $t \in [-1, 1]$ and fixed ε.



Chebyshev series approx ($\varepsilon = 25$)

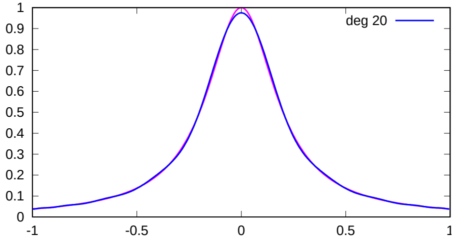
deg	$\|\cdot\|_\infty$	
10	0.17	
12	0.12	
14	0.079	
20	0.025	

Example

Compute $\boldsymbol{x}^\circ(t) \approx \dfrac{1}{1+\varepsilon t^2}$ and $r \geq \left\| \boldsymbol{x}^\circ(t) - \dfrac{1}{1+\varepsilon t^2} \right\|_\infty$ for $t \in [-1, 1]$ and fixed ε.

- Newton-like operator $\mathbf{T} \cdot x = x - \tilde{\psi}((1 + \varepsilon t^2)x - 1) \qquad \tilde{\psi} \approx 1/(1 + \varepsilon t^2)$

RPAs for the reciprocal or division



Chebyshev series approx ($\varepsilon = 25$)

deg	$\|\cdot\|_\infty$	μ	
10	0.17	0.22	
12	0.12	0.14	
14	0.079	0.088	
20	0.025	0.026	

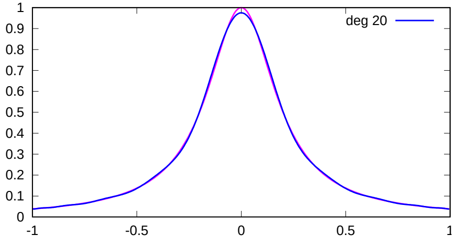
Example

Compute $\boldsymbol{x}^\circ(t) \approx \dfrac{1}{1+\varepsilon t^2}$ and $r \geq \left\| \boldsymbol{x}^\circ(t) - \dfrac{1}{1+\varepsilon t^2} \right\|_\infty$ for $t \in [-1, 1]$ and fixed ε.

- Newton-like operator $\mathbf{T} \cdot x = x - \tilde{\psi}((1+\varepsilon t^2)x - 1)$ $\qquad \tilde{\psi} \approx 1/(1+\varepsilon t^2)$
- Is $\mathbf{T}$ contracting?

$$\|\mathrm{D}\mathbf{T}\| = \|1 - \tilde{\psi}(1+\varepsilon t^2)\|_\infty = \mu < 1$$

RPAs for the reciprocal or division



Chebyshev series approx ($\varepsilon = 25$)

deg	$\|\cdot\|_\infty$	μ	b	r
10	0.17	0.22	0.16	0.20
12	0.12	0.14	0.11	0.13
14	0.079	0.088	0.076	0.083
20	0.025	0.026	0.025	0.026

Example

Compute $\boldsymbol{x}^\circ(t) \approx \dfrac{1}{1+\varepsilon t^2}$ and $r \geq \left\| \boldsymbol{x}^\circ(t) - \dfrac{1}{1+\varepsilon t^2} \right\|_\infty$ for $t \in [-1, 1]$ and fixed ε.

- Newton-like operator $\mathbf{T} \cdot x = x - \tilde{\psi}((1+\varepsilon t^2)x - 1)$ $\qquad \tilde{\psi} \approx 1/(1+\varepsilon t^2)$
- Is $\mathbf{T}$ contracting?

$$\|\mathrm{D}\mathbf{T}\| = \|1 - \tilde{\psi}(1+\varepsilon t^2)\|_\infty = \mu < 1$$

- Apply the Banach fixed-point theorem:

$$\|\boldsymbol{x}^\circ - \mathbf{T} \cdot \boldsymbol{x}^\circ\|_\infty = \|\tilde{\psi}((1+\varepsilon t^2)\boldsymbol{x}^\circ - 1)\|_\infty \leq b \quad \Rightarrow \quad \|\boldsymbol{x}^\circ - \boldsymbol{x}^\star\|_\infty \leq r := b/(1-\mu)$$

Division of two Chebyshev models

Given:

- $f, g \in \mathcal{C}([-1,1])$ represented by Chebyshev models $\boldsymbol{f} = (f^\circ, \varepsilon)$ and $\boldsymbol{g} = (g^\circ, \eta)$,
- $h^\circ \in \mathbb{R}[x]$ a polynomial approximation of $h^* = f/g$,
- $k^\circ \in \mathbb{R}[x]$ a polynomial approximation of $1/g$,

we have the following rigorous upper bound on the approximation error:

$$\|h^\circ - f/g\|_\infty \le \tau = \frac{b}{1-\mu},$$

provided that we have computed b and $\mu < 1$ such that:

$$\|1 - k^\circ g^\circ\|_\infty + \eta\|k^\circ\|_\infty \le \mu,$$
$$\|k^\circ(g^\circ h^\circ - f^\circ)\|_\infty + \eta\|k^\circ h^\circ\|_\infty + \varepsilon\|k^\circ\|_\infty \le b.$$

Hence, $\boldsymbol{h} = (h^\circ, \tau)$ is a Chebyshev model for $h^* = f/g$.

Square Root of an RPA with a Newton-like approach

- $\boldsymbol{x}^{\circ}(t) \approx \sqrt{f(t)}$ where $f(t) = 1 + \varepsilon t^2$.

Square Root of an RPA with a Newton-like approach

- $\boldsymbol{x}^{\circ}(t) \approx \sqrt{f(t)}$ where $f(t) = 1 + \varepsilon t^2$.
- $\boldsymbol{x}^{\star} = \sqrt{f}$ unique fixed point of:

$$\mathbf{T} \cdot x = x - \frac{\tilde{\psi}}{2}(x^2 - f) \qquad \tilde{\psi}(t) \approx 1/\boldsymbol{x}^{\circ}(\boldsymbol{t}) \approx 1/\sqrt{f(t)}$$

Square Root of an RPA with a Newton-like approach

- $x^\circ(t) \approx \sqrt{f(t)}$ where $f(t) = 1 + \varepsilon t^2$.
- $x^\star = \sqrt{f}$ unique fixed point of:

$$\mathbf{T} \cdot x = x - \frac{\tilde{\psi}}{2}(x^2 - f) \qquad \tilde{\psi}(t) \approx 1/\boldsymbol{x}^\circ\boldsymbol{(t)} \approx 1/\sqrt{f(t)}$$

- Is $\mathbf{T}$ contracting?

$$\|\mathrm{D}\mathbf{T}(x)\| = \|1 - \tilde{\psi} x\| \leq \|1 - \tilde{\psi}\boldsymbol{x}^\circ\| + \|\tilde{\psi}\|\|x - \boldsymbol{x}^\circ\|$$

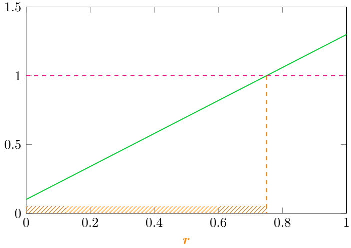
Square Root of an RPA with a Newton-like approach

- $\boldsymbol{x}^{\circ}(t) \approx \sqrt{f(t)}$ where $f(t) = 1 + \varepsilon t^2$.
- $\boldsymbol{x}^{\star} = \sqrt{f}$ unique fixed point of:

$$\mathbf{T} \cdot x = x - \frac{\tilde{\psi}}{2}(x^2 - f) \qquad \tilde{\psi}(t) \approx 1/\boldsymbol{x}^{\circ}(\boldsymbol{t}) \approx 1/\sqrt{f(t)}$$

- Is $\mathbf{T}$ contracting?

$$\boldsymbol{\mu} = \sup_{\|x - \boldsymbol{x}^{\circ}\| \leq \boldsymbol{r}} \|\mathrm{D}\mathbf{T}(x)\| \leq \|1 - \tilde{\psi}\boldsymbol{x}^{\circ}\| + \|\tilde{\psi}\| \boldsymbol{r}$$



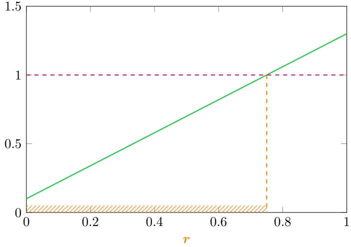
Square Root of an RPA with a Newton-like approach

- $x^\circ(t) \approx \sqrt{f(t)}$ where $f(t) = 1 + \varepsilon t^2$.
- $x^\star = \sqrt{f}$ unique fixed point of:

$$\mathbf{T} \cdot x = x - \frac{\tilde{\psi}}{2}(x^2 - f) \qquad \tilde{\psi}(t) \approx 1/\boldsymbol{x^\circ(t)} \approx 1/\sqrt{f(t)}$$

- Is $\mathbf{T}$ contracting?

$$\mu = \sup_{\|x - x^\circ\| \leq r} \|\mathrm{D}\mathbf{T}(x)\| \leq \|1 - \tilde{\psi} x^\circ\| + \|\tilde{\psi}\| r$$



- Stable neighborhood for x°:

$$\|x^\circ - \mathbf{T} \cdot x^\circ\| + \mu r \leq r$$

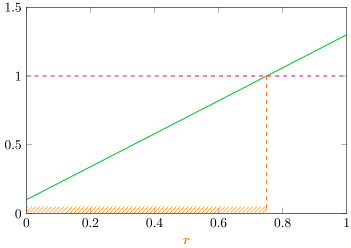
Square Root of an RPA with a Newton-like approach

- $\boldsymbol{x}^{\circ}(t) \approx \sqrt{f(t)}$ where $f(t) = 1 + \varepsilon t^2$.
- $\boldsymbol{x}^{\star} = \sqrt{f}$ unique fixed point of:

$$\mathbf{T} \cdot x = x - \frac{\tilde{\psi}}{2}(x^2 - f) \qquad \tilde{\psi}(t) \approx 1/\boldsymbol{x}^{\circ}(\boldsymbol{t}) \approx 1/\sqrt{f(t)}$$

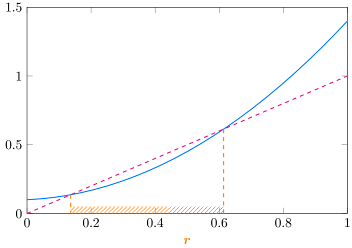
- Is $\mathbf{T}$ contracting?

$$\boldsymbol{\mu} = \sup_{\|x - \boldsymbol{x}^{\circ}\| \leq \boldsymbol{r}} \|\mathrm{D}\mathbf{T}(x)\| \leq \|1 - \tilde{\psi}\boldsymbol{x}^{\circ}\| + \|\tilde{\psi}\|\boldsymbol{r}$$



- Stable neighborhood for $\boldsymbol{x}^{\circ}$:

$$\|\tilde{\psi}(\boldsymbol{x}^{\circ 2} - f)/2\| + \boldsymbol{r}(\|1 - \tilde{\psi}\boldsymbol{x}^{\circ}\| + \|\tilde{\psi}\|\boldsymbol{r}) \leq \boldsymbol{r}$$



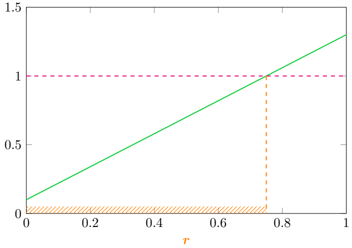
Square Root of an RPA with a Newton-like approach

- $x^\circ(t) \approx \sqrt{f(t)}$ where $f(t) = 1 + \varepsilon t^2$.
- $x^\star = \sqrt{f}$ unique fixed point of:

$$\mathbf{T} \cdot x = x - \frac{\tilde{\psi}}{2}(x^2 - f) \qquad \tilde{\psi}(t) \approx 1/x^\circ(t) \approx 1/\sqrt{f(t)}$$

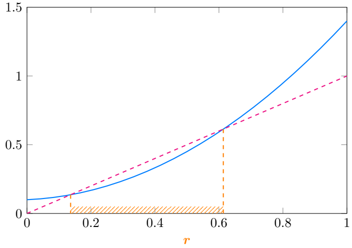
- Is $\mathbf{T}$ contracting?

$$\mu = \sup_{\|x - x^\circ\| \leq r} \|\mathrm{D}\mathbf{T}(x)\| \leq \|1 - \tilde{\psi} x^\circ\| + \|\tilde{\psi}\| r$$



- Stable neighborhood for x°:

$$\|\tilde{\psi}(x^{\circ 2} - f)/2\| + r(\|1 - \tilde{\psi} x^\circ\| + \|\tilde{\psi}\| r) \leq r$$



- Apply the Banach fixed-point theorem!

Square root of a Chebyshev model

Given:

- $f \in \mathcal{C}([-1,1])$ represented by a Chebyshev model $\boldsymbol{f} = (f^\circ, \varepsilon)$,
- $g^\circ \in \mathbb{R}[x]$ a polynomial approximation of $g^* = \sqrt{f}$,
- $k^\circ \in \mathbb{R}[x]$ a polynomial approximation of $1/g^\circ$,

we have the following rigorous upper bound on the approximation error:

$$\|g^\circ - \sqrt{f}\|_\infty \leq \eta = \frac{\eta'}{1-\mu},$$

provided that we have computed $\mu_0, \mu_1, \eta', \Delta, r^\circ, \mu$ satisfying:

$$\|1 - k^\circ g^\circ\|_\infty \leq \mu_0 < 1, \; \|k^\circ\|_\infty \leq \mu_1, \; \|k^\circ({g^\circ}^2 - f^\circ)\|_\infty + \varepsilon \|k^\circ\|_\infty \leq 2\eta',$$

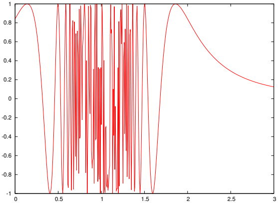
$$\Delta := (1-\mu_0)^2 - 4\mu_1\eta' \geq 0, \qquad r^\circ := \frac{1-\mu_0-\sqrt{\Delta}}{2\mu_1},$$

$$\mu := \mu_0 + \mu_1 r^\circ < 1.$$

Hence, $\boldsymbol{g} = (g^\circ, \eta)$ is a Chebyshev model for $g^* = \sqrt{f}$.

Quadrature: an example

Let $J = \int_0^3 \sin\left(\frac{1}{(10^{-3} + (1-x)^2)^{3/2}}\right) \mathrm{d}x.$

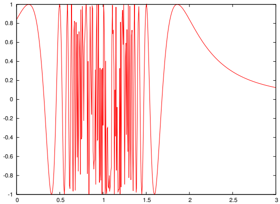


- Chen, '06: 0.7578918118.

WHAT IS THE CORRECT ANSWER?

Using Chebyshev-based RPAs*: $0.749974368527[1,3]$.

Let $J = \int_0^3 \sin\left(\frac{1}{(10^{-3} + (1-x)^2)^{3/2}}\right) \mathrm{d}x$.



- Chen, '06: 0.7578918118.

WHAT IS THE CORRECT ANSWER?

Using Chebyshev-based RPAs*: $0.749974368527[1, 3]$.

*N. Brisebarre, M.J., *Chebyshev interpolation polynomial-based tools for rigorous computing*, ISSAC2010

Few remarks about basic ODE validation

Rough enclosures of IVP differential equations

$$u'(t) = f(t, u(t)),$$
$$u(t_0) = u_0,\ u_0 \in U_0,\ t \in [0, T]$$

with $f \in \mathcal{C}([0, T] \times \mathbb{R})$, Lipschitz-continuous in the second variable (uniformly in t):

$\exists L > 0$ s.t $|f(t, x) - f(t, y)| \leq L|x - y|$ for all (t, x), $(t, y) \in [0, T] \times \mathbb{R}$.

Verification condition

If there exist $0 < h \leq T$ and $U_h \in \mathbb{IR}$, $U_0 \subseteq U_h$ s.t.

$$U_0 + [0, h] f([0, h], U_h) \subset U_h,$$

then the IVP has a unique solution $u_{u_0} \in \mathcal{C}^1([0, h])$ for each $u_0 \in U_0$.

Rough enclosures of IVP differential equations

Proof sketch:

- Integral fixed-point reformulation: $\mathbf{T} : \mathcal{C}([0,h]) \to \mathcal{C}([0,h])$

$$\mathbf{T}u(t) := u_0 + \int_0^t f(s, u(s))\mathrm{d}s,\ t \in [0,h].$$

$$u = \mathbf{T}u.$$

- Check Banach fixed point hypotheses:
  - $\mathbf{T}$ is a contraction on $\mathcal{C}([0,h])$ w.r.t. the norm:

$$\|u\|_1 = \max_{0 \le t \le h} e^{-Lt}|u(t)|;$$

  - The set $X := \{u \in \mathcal{C}([0,h]) :\ u([0,h]) \subseteq U_h\}$ is closed and bounded in $(\mathcal{C}([0,h]), \|\cdot\|_1)$;
  - If

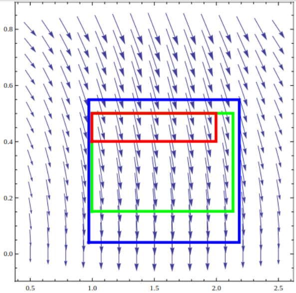
$$U_0 + [0,h] f([0,h], U_h) \subset U_h$$

    then $\mathbf{T}X \subseteq X$ for each $u_0 \in U_0$.

* Both $(\mathcal{C}([0,h]), \|\cdot\|_1)$ and $(\mathcal{C}([0,h]), \|\cdot\|_\infty)$ are Banach spaces and the norms are equivalent since: $e^{-Lh}\|u\|_\infty \le \|u\|_1 \le \|u\|_\infty$, for all $u \in \mathcal{C}([0,h])$

Example

$$x'' = -\sin(x) + 0.1x', \qquad h = 0.25$$



$$
\begin{aligned}
[\mathbf{X}] &= [1,2] \times [0.4, 0.5] \\
[\mathbf{Y}] &= [\mathbf{X}] + h[-.2, 1.5] * f([\mathbf{X}]) \subset [0.9749, 2.1875] \times [0.04, 0.548] \\
[\mathbf{Z}] &= [\mathbf{X}] + [0, h] * f([\mathbf{Y}]) \subset [1.0, 2.137] \times [0.1502, 0.5] \subset \mathrm{int}([\mathbf{Y}])
\end{aligned}
$$

* Courtesy of D. Wilczak, http://ww2.ii.uj.edu.pl/~wilczak/capd-tutorial/CAPD_tutorial_part_I.pdf

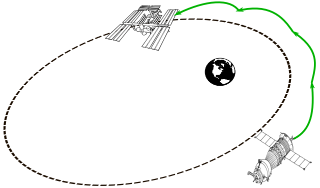
⇝ Trajectories for Linearized Impulsive Spacecraft Rendezvous Problem

Lin. Keplerian Motion [Tschauner-Hempel]

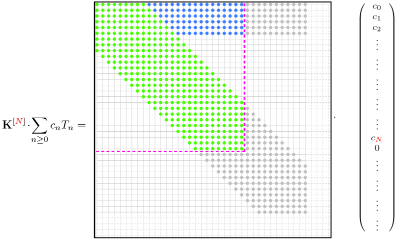
$$z''(\nu) + 2x'(\nu) - \frac{3}{1 + e\cos\nu} z(\nu) = 0$$

$$x''(\nu) - 2z'(\nu) = 0$$

$$y''(\nu) + y(\nu) = 0$$

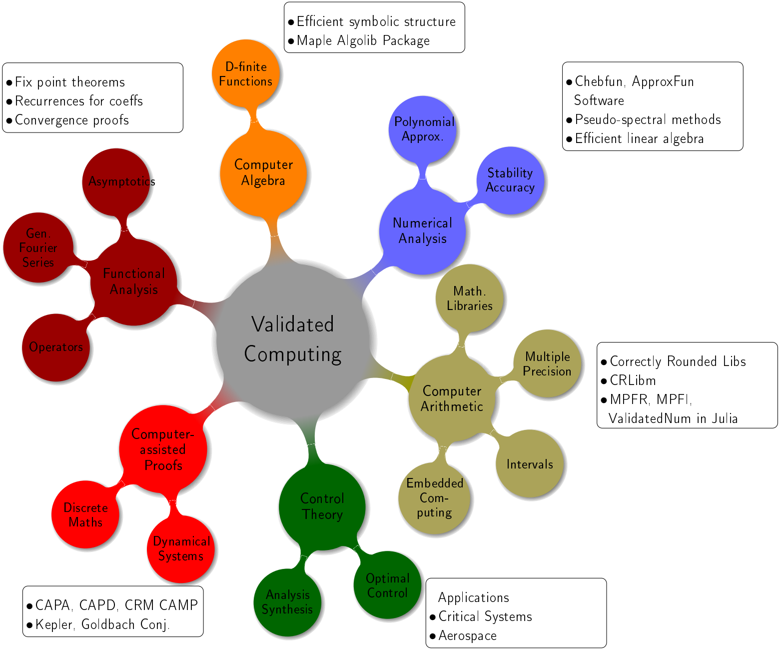


⇒ Efficient spectral methods based on truncated Chebyshev series with a posteriori validation



- General LODEs (nonpolynomial coefficients); Coupled systems of LODEs

*F. Bréhard, N. Brisebarre, and M. J., *Validated and numerically efficient Chebyshev spectral methods for linear ordinary differential equations*, ACM TOMS, 2018


Validated Computing
Computer Algebra
D-finite Functions
• Efficient symbolic structure
• Maple Algolib Package
Numerical Analysis
Polynomial Approx.
Stability Accuracy
• Chebfun, ApproxFun Software
• Pseudo-spectral methods
• Efficient linear algebra
Computer Arithmetic
Math. Libraries
Multiple Precision
Intervals
Embedded Com-puting
• Correctly Rounded Libs
• CRLibm
• MPFR, MPFI, ValidatedNum in Julia
Control Theory
Analysis Synthesis
Optimal Control
Applications
• Critical Systems
• Aerospace
Computer-assisted Proofs
Discrete Maths
Dynamical Systems
• CAPA, CAPD, CRM CAMP
• Kepler, Goldbach Conj.
Functional Analysis
Asymptotics
Gen. Fourier Series
Operators
• Fix point theorems
• Recurrences for coeffs
• Convergence proofs

Outlets for Rigorous Computing

- Solve problems with the required accuracy
  $\rightarrow$ guarantees/proofs about results.

Outlets for Rigorous Computing

- Solve problems with the required accuracy
  $\rightarrow$ guarantees/proofs about results.

- Modern computer differential algebra algorithms + approximation in suitable functional spaces
  $\rightsquigarrow$ obtain efficient approximations and analytic error bounds
  $\rightsquigarrow$ Theoretical and practical complexity issues for generic classes of problems

Outlets for Rigorous Computing

- Solve problems with the required accuracy
  $\to$ guarantees/proofs about results.

- Modern computer differential algebra algorithms + approximation in suitable functional spaces
  $\rightsquigarrow$ obtain efficient approximations and analytic error bounds
  $\rightsquigarrow$ Theoretical and practical complexity issues for generic classes of problems

- CAP for Dynamical Systems

Outlets for Rigorous Computing

- Solve problems with the required accuracy
  $\rightarrow$ guarantees/proofs about results.

- Modern computer differential algebra algorithms + approximation in suitable functional spaces
  $\rightsquigarrow$ obtain efficient approximations and analytic error bounds
  $\rightsquigarrow$ Theoretical and practical complexity issues for generic classes of problems

- CAP for Dynamical Systems

- Interaction with Optimization in the framework of Optimal Control
  Aerospace Applications

Outlets for Rigorous Computing

- Solve problems with the required accuracy
  $\to$ guarantees/proofs about results.

- Modern computer differential algebra algorithms + approximation in suitable functional spaces
  $\leadsto$ obtain efficient approximations and analytic error bounds
  $\leadsto$ Theoretical and practical complexity issues for generic classes of problems

- CAP for Dynamical Systems

- Interaction with Optimization in the framework of Optimal Control
  Aerospace Applications

- Formal proofs of the above

Thank you for your attention!