Approximation Theory and Proof Assistants: Certified Computations

Nicolas Brisebarre and Damien Pous

Master 2 Informatique Fondamentale
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We have $\deg T_n = \deg U_n = n$ for all $n \in \mathbb{N}$.

Therefore, $(T_k)_{0 \leqslant k \leqslant n}$ is a basis of $\mathbb{R}_n[x]$.

Now, we give results that allow for (fast) computing the coefficients of interpolation polynomials, at the Chebyshev nodes, expressed in the basis $(T_k)_{0 \leqslant k \leqslant n}$.

Proposition

(Discrete orthogonality.)

1. *We have*

$$\sum_{k=0}^{n} T_i(\mu_k)\, T_j(\mu_k) = \begin{cases} 0, & i \neq j, \\ n+1, & i = j = 0, \\ \frac{n+1}{2}, & i = j \neq 0. \end{cases}$$

2. *We have*

$$\sum_{k=0}^{n} T_i(\nu_k)\, T_j(\nu_k) = \begin{cases} 0, & i \neq j, \\ n, & i = j \in \{0, n\}, \\ \frac{n}{2}, & i = j \notin \{0, n\}. \end{cases}$$

Section 2.4. Interpolation and approximation, Chebyshev polynomials

$\sum'$ denotes that the first term of the sum has to be halved, $\sum''$ denotes that the first and the last terms of the sum have to be halved.

Proposition

1. *If* $p_{1,n} = \sum'_{0 \leqslant i \leqslant n} c_{1,i} T_i \in \mathbb{R}_n[x]$ *interpolates* f *on the set* $\{\mu_k : 0 \leqslant k \leqslant n\}$, *then*

$$c_{1,i} = \frac{2}{n+1} \sum_{k=0}^{n} f(\mu_k) T_i(\mu_k).$$

2. *Likewise, if* $p_{2,n} = \sum''_{0 \leqslant i \leqslant n} c_{2,i} T_i$ *interpolates* f *at* $\{\nu_k : 0 \leqslant k \leqslant n\}$, *then*

$$c_{2,i} = \frac{2}{n} \sum_{k=0}^{n} f(\nu_k) T_i(\nu_k).$$

Cost: $O(n^2)$ operations.

Section 2.5. Clenshaw's method for evaluating Chebyshev sums

Given coefficients $c_0, \ldots, c_N$ and a point t, we would like to compute the sum

$$\sum_{k=0}^{N} c_k T_k\left(t\right).$$

Recall that the polynomials T_k satisfy $T_{k+2}\left(x\right) = 2xT_{k+1}\left(x\right) - T_k\left(x\right)$.

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Method is numerically unstable.
The $U_k(x)$ satisfy the same recurrence but grows faster: we have

$$\|T_k\|_\infty = 1, \qquad \|U_k\|_\infty = k+1.$$

Section 2.5. Clenshaw's method for evaluating Chebyshev sums

Algorithm 2

Input Chebyshev coefficients $c_0, \ldots, c_N$, a point t

Output $\sum_{k=0}^{N} c_k T_k(t)$

1. $b_{N+1} \leftarrow 0$, $b_N \leftarrow c_N$
2. for $k = N-1, N-2, \ldots, 1$
    1. $b_k \leftarrow 2tb_{k+1} - b_{k+2} + c_k$
3. return $c_0 + tb_1 - b_2$

This algorithm runs in $O(N)$ arithmetic operations.

Section 2.6. Computation of the Chebyshev coefficients

How do we compute the c_k?
Assume we use the Chebyshev nodes of the second kind and obtain the result on the Chebyshev basis.

Given $y_0, \ldots, y_N$, we are looking for $c_0, \ldots, c_N$ such that $p(x) = \sum_{j=0}^{N}{}'' c_j T_j(x)$ satisfies $p(\nu_k) = y_k$ for all k.
By discrete orthogonality, we have

$$c_j = \frac{2}{N} \sum_{k=0}^{N} y_k T_k(\nu_j).$$

Observe that we have

$$T_k(\nu_j) = \cos\left(jk\frac{\pi}{N}\right)$$

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DCT: Discrete Cosine Transform. JPEG, MPEG, MP3, etc.

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Chapter 2. Orthogonal Polynomials - Chebyshev series

Section 2.1. Orthogonal Polynomials

Let $(a, b) \subset \mathbb{R}$ be an open interval, and let w be a weight function, that is to say $w : (a, b) \to (0, \infty)$ is a continuous function. We assume

$$\forall n \in \mathbb{N}, \quad \int_a^b |x|^n w(x) \mathrm{d}x < \infty.$$

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$$\forall n \in \mathbb{N}, \quad \int_a^b |x|^n w(x) \mathrm{d}x < \infty.$$

This is the case, for instance, if (a, b) is bounded and

$$\int_a^b w(x) \mathrm{d}x < \infty.$$

Section 2.1. Orthogonal Polynomials

Let

$$\mathcal{E}(w) = \left\{ f \in \mathcal{C}((a,b)) : \|f\|_2 := \left(\int_a^b f(x)^2 w(x) \mathrm{d}x \right)^{1/2} < \infty \right\}.$$

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Observe that $\mathbb{R}[x] \subset \mathcal{E}(w)$. The space $\mathcal{E}(w)$ is equipped with an inner product

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x) \mathrm{d}x;$$

and $\| \cdot \|_2$ is the norm associated to this inner product.

Section 2.1. Orthogonal Polynomials

Definition 1

A family of orthogonal polynomials associated with w is a sequence $(p_n) \in \mathbb{R}[x]^{\mathbb{N}}$ where $\deg p_k = k$ for all k, and

$$i \neq j \quad \Rightarrow \quad \langle p_i, p_j \rangle = 0.$$

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Gram-Schmidt orthogonalization process

Theorem 3

The polynomials $(p_n)_{n\in\mathbb{N}}$ *satisfy the recurrence relation*

$$p_n(x) = (x - \alpha_n)p_{n-1}(x) - \beta_n p_{n-2}(x) \qquad (n \geqslant 2)$$

with

$$\alpha_n = \frac{\langle x p_{n-1}, p_{n-1}\rangle}{\|p_{n-1}\|_2^2}, \qquad \beta_n = \frac{\|p_{n-1}\|_2^2}{\|p_{n-2}\|_2^2}.$$

Section 2.1. Orthogonal Polynomials

Example 4

$(-1,1)$	$w(x) = (1-x^2)^{-1/2}$	Chebyshev polynomials of the first kind (up to normalization)
$(-1,1)$	$w(x) = 1$	Legendre polynomials
$(0,+\infty)$	$w(x) = e^{-x}$	Laguerre polynomials
$(-\infty,\infty)$	$w(x) = e^{-x^2}$	Hermite polynomials

Exercise

Prove that the first statement of Example 4 is correct.

Theorem 5

For any weight w and for all n, the polynomial p_n has n distinct zeros in (a, b).

Theorem 6

Let $f \in \mathcal{E}(w)$, $n \in \mathbb{N}$. There exists a unique best $L_2(w)$ polynomial approximation in $\mathbb{R}_n[x]$to f, denoted $p_{2,n}$:

$$\|f - p_{2,n}\|_2 = \min_{p \in \mathbb{R}_n[x]} \|f - p\|_2.$$

It is characterized by

$$\forall p \in \mathbb{R}_n[x], \quad \langle f - p_{2,n}, p \rangle = 0.$$

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Exercise

Prove the previous theorem.

Theorem 7

If (a,b) is bounded, then for all $f \in \mathcal{E}(w)$, we have $p_{2,n} \xrightarrow{\|\cdot\|_2} f$ as $n \to \infty$.

Section 2.2. A little bit of quadrature: Gauss methods

Let w be a weight function over (a, b), and let $f \in \mathcal{C}([a, b])$. We briefly study methods which approximate the integral

$$\int_a^b f(x)w(x)\mathrm{d}x$$

with a sum of the form

$$\sum_{k=0}^{n} w_k f(x_k), \qquad w_k \in \mathbb{R}, \quad x_k \in [a, b] \text{ pairwise distinct.}$$

Section 2.2. A little bit of quadrature: Gauss methods

First of all, if $\ell_k(x) = \prod_{\substack{j=0,\\ j\neq k}}^{n} \frac{x - x_j}{x_k - x_j}$, observe that if

$$p(x) = \sum_{k=0}^{n} f(x_k)\ell_k(x) \in \mathbb{R}_n[x]$$

interpolates f at the points $x_0, \ldots, x_n$, then our approximation for the integral is equal to $\int_a^b p(x)w(x)\mathrm{d}x = \sum_{k=0}^{n} w_k f(x_k)$ with

$$w_k = \int_a^b \ell_k(x)w(x)\mathrm{d}x \text{ for } k = 0, \ldots, n.$$

Section 2.2. A little bit of quadrature: Gauss methods

Theorem 8

There exists a unique choice of the points x_k and the weights w_k such that, whenever $f \in \mathbb{R}_{2n+1}[x]$,

$$\int_a^b f(x)w(x)\mathrm{d}x = \sum_{k=0}^{n} w_k f(x_k).$$

These points x_k belong to (a,b) and are the roots of the $(n+1)$-th orthogonal polynomial associated to w.