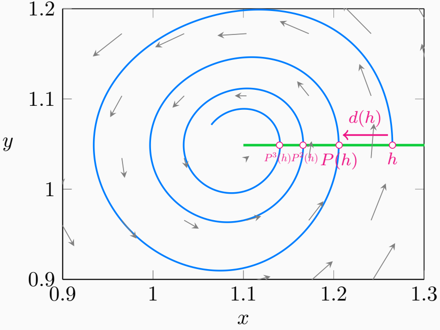


- Poincaré first return map $P(h)$
- Displacement $d(h) = P(h) - h$
- Limit cycle $\Leftrightarrow$ isolated zero of d
- Abelian integral $\mathcal{I}(h)$:

$$\oint_{H^{-1}(h)} f(x,y)\mathrm{d}y - g(x,y)\mathrm{d}x$$

$$\begin{cases} \dot{x} = -\partial_y H(x,y) + \varepsilon f(x,y) \\ \dot{y} = \partial_x H(x,y) + \varepsilon g(x,y) \end{cases}$$

Poincaré-Pontryagin theorem

The Abelian integral $\mathcal{I}(h)$ approximates the displacement function $d(h)$ for small ε:

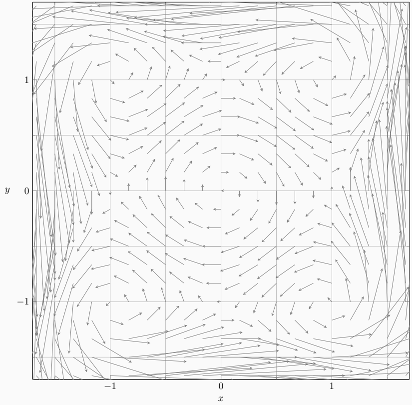
$$d(h) = \varepsilon(\mathcal{I}(h) + O(\varepsilon)) \qquad \text{when } \varepsilon \to 0$$

limit cycles $\equiv$ changes of sign of $\mathcal{I}(h)$ $\equiv$ simple zeros of $\mathcal{I}(h)$

A Pseudo-Hamiltonian Quartic System

- Hamiltonian system:

$$\begin{cases} \dot{x} = -4y(y^2 - 1.1) \\ \dot{y} = 4x(x^2 - 0.9) \end{cases}$$



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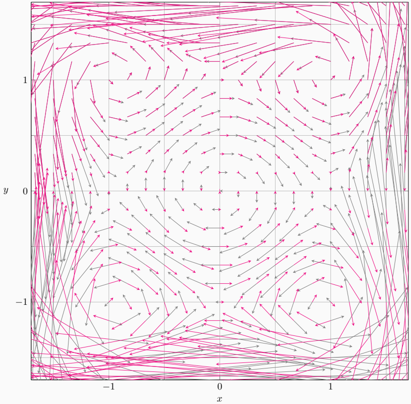

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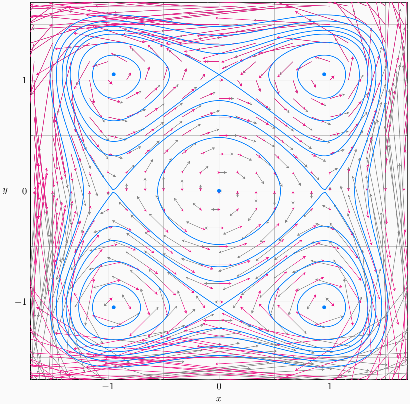


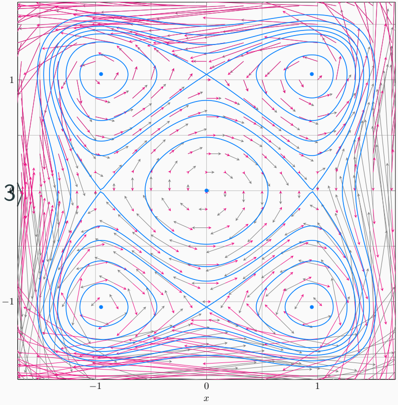
A Pseudo-Hamiltonian Quartic System

- pseudo-Hamiltonian system:

$$\begin{cases} \dot{x} = -4yy(y^2 - 1.1) + \varepsilon f(x, y) \\ \dot{y} = 4yx(x^2 - 0.9) + \varepsilon g(x, y) \end{cases}$$

- same geometric orbits after rescaling
- $\simeq$ perturbations without rescaling:

$$\frac{f(x,y)}{y}, \frac{g(x,y)}{y} \in \langle x^i y^j, i \geq 0, j \geq -1, i+j \leq 3 \rangle$$



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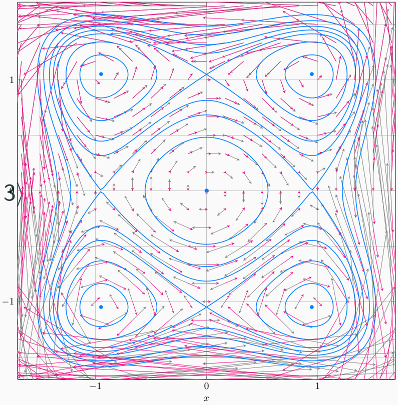

Generalized Poincaré-Pontryagin theorem

The *generalized* Abelian integral:

$$\mathcal{I}(h) = \oint_{H^{-1}(h)} \frac{f(x, y)\mathrm{d}y - g(x, y)\mathrm{d}x}{y}$$

approximates the displacement function $d(h)$ for small ε:

$$d(h) = \varepsilon(\mathcal{I}(h) + O(\varepsilon)) \qquad \text{when } \varepsilon \to 0$$

A Pseudo-Hamiltonian Quartic System

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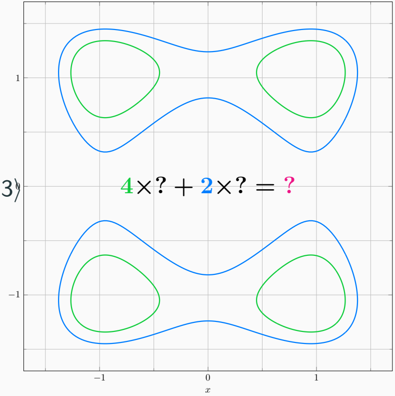

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approximates the displacement function $d(h)$ for small ε:

$$d(h) = \varepsilon(\mathcal{I}(h) + O(\varepsilon)) \qquad \text{when } \varepsilon \to 0$$

$\Rightarrow$ Maximize the number of zeros of $\mathcal{I}(h)$

Challenge for Abelian Integrals

polynomial or rational functions

$$\int_{\Gamma(h)} f(x,y)\mathrm{d}y - g(x,y)\mathrm{d}x$$

oval $H(x,y)=h$ with H polynomial or rational

polynomial or rational functions

$$\int_{\Gamma(h)} f(x,y)\mathrm{d}y - g(x,y)\mathrm{d}x$$

oval $H(x,y) = h$ with H polynomial or rational

GOALS:

- Find the coefficients of f and g
- High accuracy evaluation in good complexity
- Rigorous and tight error bounds

polynomial or rational functions

$$\int_{\Gamma(h)} f(x, y)\mathrm{d}y - g(x, y)\mathrm{d}x$$

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GOALS:

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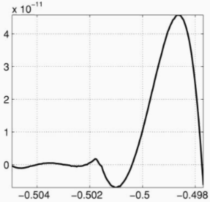
- High accuracy evaluation in good complexity
- Rigorous and tight error bounds
- Certified implementation in a proof assistant
  e.g., T. Johnson, 2011: $\mathcal{H}(4) \geq 26$



Credit: T. Johnson

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$$\int_{\Gamma(h)} f(x,y)\,\mathrm{d}y - g(x,y)\,\mathrm{d}x$$

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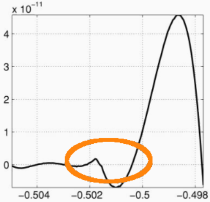

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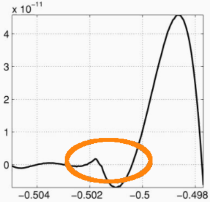
  (Bréhard, Brisebarre, Joldes, Tucker)



Credit: T. Johnson

Choice of Perturbations

$$
\begin{array}{rcccccc}
f(x,y) & = & 1 & x & y & x^2 & xy \\
 & & y^2 & x^3 & x^2y & xy^2 & y^3 \\
 & & x^4 & x^3y & x^2y^2 & xy^3 & y^4
\end{array}
$$

$$
\begin{array}{rcccccc}
g(x,y) & = & 1 & x & y & x^2 & xy \\
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- integrable perturbations: $\partial_x \frac{f(x,y)}{y} = 0$, $\partial_y \frac{g(x,y)}{y} = 0$.

$$
\begin{array}{rclllll}
f(x,y) & = & \color{lightgray}{1} & x & \color{lightgray}{y} & x^2 & xy \\
 & & \color{lightgray}{y^2} & x^3 & x^2y & xy^2 & \color{lightgray}{y^3} \\
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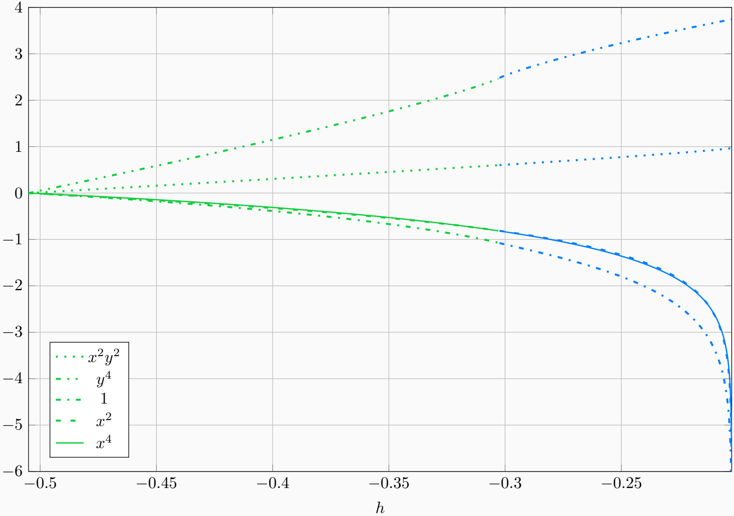
$$\mathcal{I}(h) = \oint_{H^{-1}(h)} \frac{\alpha_{00} + \alpha_{20}x^2 + \alpha_{22}x^2y^2 + \alpha_{40}x^4 + \alpha_{04}y^4}{y} \mathrm{d}x$$

Numerically Optimizing the Number of Zeros

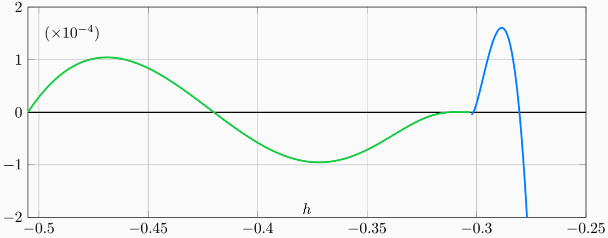
- Find coefficients of $\mathcal{I}(h) = \alpha_{00}\mathcal{I}_{00}(h) + \alpha_{20}\mathcal{I}_{20}(h) + \alpha_{22}\mathcal{I}_{22}(h) + \alpha_{40}\mathcal{I}_{40}(h) + \alpha_{04}\mathcal{I}_{04}(h$



$$\mathcal{I}_{ij}(h) = \oint_{H^{-1}(h)} \frac{x^i y^j}{y} \mathrm{d}x$$

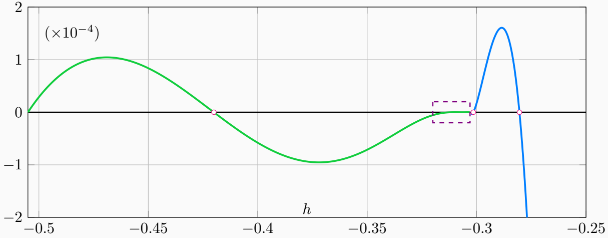
Numerically Optimizing the Number of Zeros

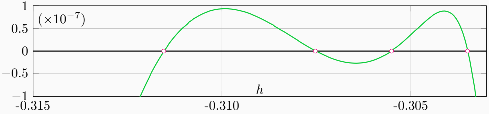
α_{00}	$=$	-0.786221486678548376644
α_{20}	$=$	0.877235236126534360511
α_{22}	$=$	1
α_{40}	$=$	0.237427138942930382233
α_{04}	$=$	-0.218238461730788637533



Numerically Optimizing the Number of Zeros

α_{00}	$=$	-0.78622148667854837664
α_{20}	$=$	0.87723523612653436051
α_{22}	$=$	1
α_{40}	$=$	0.23742713894293038223
α_{04}	$=$	-0.21823846173078863753





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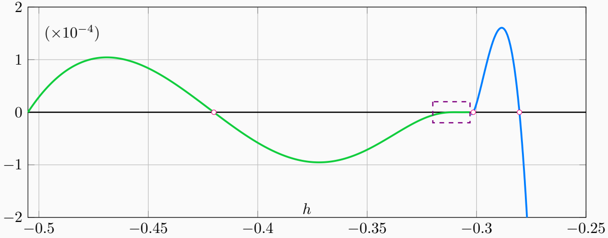
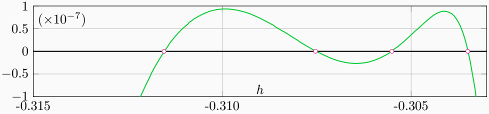
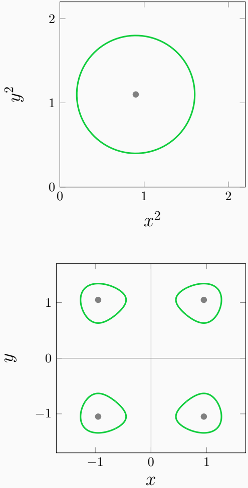




$4 \times 5 + 2 \times 2 = 24$

Computing Abelian Integrals

$H(x, y) = (x^2 - 0.9)^2 + (y^2 - 1.1)^2$ $\qquad 0 < \sqrt{h} =: r < 0.9$



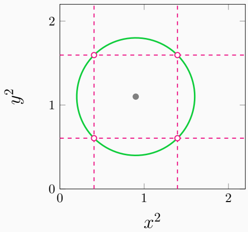
Computing Abelian Integrals

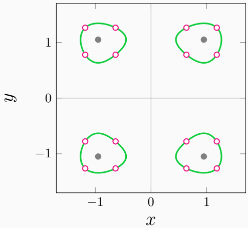
$$H(x,y) = (x^2 - 0.9)^2 + (y^2 - 1.1)^2$$

$$0 < \sqrt{h} =: r < 0.9$$

$$x_{\mathrm{min}} = \sqrt{0.9 - \frac{r}{\sqrt{2}}} \qquad x_{\mathrm{max}} = \sqrt{0.9 + \frac{r}{\sqrt{2}}}$$

$$y_{\mathrm{min}} = \sqrt{1.1 - \frac{r}{\sqrt{2}}} \qquad y_{\mathrm{max}} = \sqrt{1.1 + \frac{r}{\sqrt{2}}}$$





Computing Abelian Integrals

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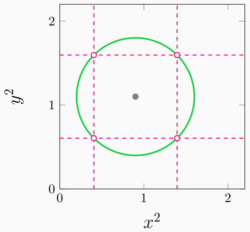
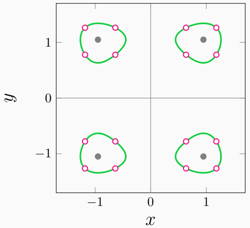




$$x_{\text{min}} = \sqrt{0.9 - \frac{r}{\sqrt{2}}} \qquad x_{\text{max}} = \sqrt{0.9 + \frac{r}{\sqrt{2}}}$$

$$y_{\text{min}} = \sqrt{1.1 - \frac{r}{\sqrt{2}}} \qquad y_{\text{max}} = \sqrt{1.1 + \frac{r}{\sqrt{2}}}$$

$$y_{\text{up}}(x) = \sqrt{1.1 + \sqrt{r^2 - (x^2 - 0.9)^2}}$$

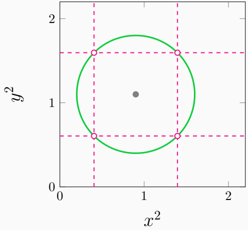
$$y_{\text{down}}(x) = \sqrt{1.1 - \sqrt{r^2 - (x^2 - 0.9)^2}}$$

$$x_{\text{left}}(y) = \sqrt{0.9 - \sqrt{r^2 - (y^2 - 1.1)^2}}$$

$$x_{\text{right}}(y) = \sqrt{0.9 + \sqrt{r^2 - (y^2 - 1.1)^2}}$$

Computing Abelian Integrals

$$H(x,y) = (x^2-0.9)^2+(y^2-1.1)^2 \qquad 0 < \sqrt{h} =: r < 0.9$$



$$x_{\text{min}} = \sqrt{0.9 - \frac{r}{\sqrt{2}}} \qquad x_{\text{max}} = \sqrt{0.9 + \frac{r}{\sqrt{2}}}$$

$$y_{\text{min}} = \sqrt{1.1 - \frac{r}{\sqrt{2}}} \qquad y_{\text{max}} = \sqrt{1.1 + \frac{r}{\sqrt{2}}}$$

$$y_{\text{up}}(x) = \sqrt{1.1 + \sqrt{r^2 - (x^2-0.9)^2}}$$

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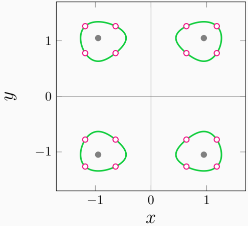
$$x_{\text{left}}(y) = \sqrt{0.9 - \sqrt{r^2 - (y^2-1.1)^2}}$$

$$x_{\text{right}}(y) = \sqrt{0.9 + \sqrt{r^2 - (y^2-1.1)^2}}$$



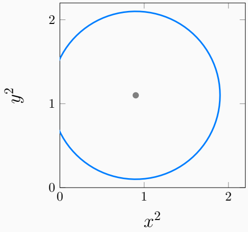
$$\mathcal{I}(h) = \oint_{H^{-1}(h)} \frac{g(x,y)}{y}\,\mathrm{d}x = \int_{x_{\text{min}}}^{x_{\text{max}}} \left(\frac{g(x, y_{\text{up}}(x))}{y_{\text{up}}(x)} - \frac{g(x, y_{\text{down}}(x))}{y_{\text{down}}(x)} \right) \mathrm{d}x$$

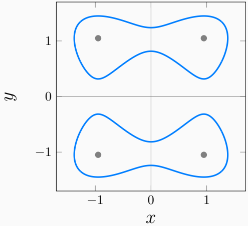
$$+ \int_{y_{\text{min}}}^{y_{\text{max}}} \left(\frac{g(x_{\text{left}}(y), y)}{x_{\text{left}}(y)} + \frac{g(x_{\text{right}}(y), y)}{x_{\text{right}}(y)} \right) \frac{y^2-1.1}{\sqrt{r^2-(y^2-1.1)^2}}\,\mathrm{d}y.$$

Computing Abelian Integrals

$H(x, y) = (x^2 - 0.9)^2 + (y^2 - 1.1)^2$

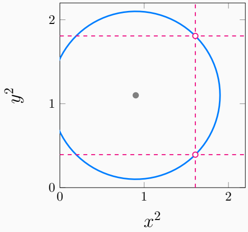
$0.9 < \sqrt{h} =: r < 1.1$





Computing Abelian Integrals

$$H(x,y) = (x^2 - 0.9)^2 + (y^2 - 1.1)^2 \qquad 0.9 < \sqrt{h} =: r < 1.1$$



$$x_{\text{min}} = 0 \qquad x_{\text{max}} = \sqrt{0.9 + \frac{r}{\sqrt{2}}}$$

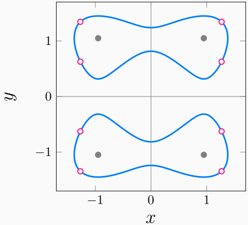
$$y_{\text{min}} = \sqrt{1.1 - \frac{r}{\sqrt{2}}} \qquad y_{\text{max}} = \sqrt{1.1 + \frac{r}{\sqrt{2}}}$$

$$y_{\text{up}}(x) = \sqrt{1.1 + \sqrt{r^2 - (x^2 - 0.9)^2}}$$

$$y_{\text{down}}(x) = \sqrt{1.1 - \sqrt{r^2 - (x^2 - 0.9)^2}}$$

$$x_{\text{left}}(y) = \sqrt{0.9 - \sqrt{r^2 - (y^2 - 1.1)^2}}$$

$$x_{\text{right}}(y) = \sqrt{0.9 + \sqrt{r^2 - (y^2 - 1.1)^2}}$$



$$\mathcal{I}(h) = \oint_{H^{-1}(h)} \frac{g(x,y)}{y}\,\mathrm{d}x = \int_{-x_{\text{max}}}^{x_{\text{max}}} \left(\frac{g(x, y_{\text{up}}(x))}{y_{\text{up}}(x)} - \frac{g(x, y_{\text{down}}(x))}{y_{\text{down}}(x)} \right) \mathrm{d}x$$

$$+ 2 \int_{y_{\text{min}}}^{y_{\text{max}}} \frac{g(x_{\text{right}}(y), y)(y^2 - 1.1)}{x_{\text{right}}(y)\sqrt{r^2 - (y^2 - 1.1)^2}}\,\mathrm{d}y.$$

A bit of Coq code